Closed Form PDF for Merton's Jump Diffusion Model

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Abstract

A closed form for the probability density function (PDF) is given for Merton's Jump Diffusion Model.

If a random variable X has a probability density function (PDF) $q(x)$ then the fourier transform of q is the associated characteristic function $\phi_q(z)$. Thus we have

$$\phi_q(z) = \int_{-\infty}^{\infty} q(x) e^{izx} dx \text{ and } q(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_q(z) e^{-izx} dz.$$

It is known (c.f. [1]) that Merton's jump diffusion model has characteristic function

$$\phi_q(z) = e^{iz\omega T - z^2 T\sigma^2/2 + (e^{iz\mu_J - z^2\sigma_J^2/2} - 1)\lambda T}$$

(valid on the entire complex plane) where

- μ_J = mean of jump process
- σ_J = standard deviation of jump process
- $\omega = r - \sigma^2/2 - \lambda^Q \int (e^x - 1) g(x) dx$, that is, $\omega = r - \sigma^2/2 - \lambda^Q \kappa$ where r is the risk-free rate of return, κ is an expected value for the jump process, and λ^Q is the intensity.

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Therefore the PDF for Merton's jump diffusion model is given by

$$\begin{aligned} q(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\phi_q(z)e^{-izx}dz \\ &= \frac{e^{-\lambda T}}{2\pi}\int_{-\infty}^{\infty} e^{iz\omega T - z^2T\sigma^2/2 + e^{iz\mu_J - z^2\sigma_J^2/2}\lambda T}\cdot e^{-izx}dz \\ &= \frac{e^{-\lambda T}}{2\pi}\int_{-\infty}^{\infty} e^{iz\omega T - izx - z^2T\sigma^2/2}e^{e^{iz\mu_J - z^2\sigma_J^2/2}\lambda T}dz \\ &= \frac{e^{-\lambda T}}{2\pi}\int_{-\infty}^{\infty} e^{iz\omega T - izx - z^2T\sigma^2/2}\sum_{n=0}^{\infty}\frac{(\lambda T)^n}{n!}(e^{iz\mu_J - z^2\sigma_J^2/2})^n dz \\ &= \frac{e^{-\lambda T}}{2\pi}\sum_{n=0}^{\infty}\frac{(\lambda T)^n}{n!}\cdot G_n(x) \qquad (1) \end{aligned}$$

where

$$G_n(x) = \int_{-\infty}^{\infty} e^{iz\omega T - izx - z^2T\sigma^2/2 + niz\mu_J - nz^2\sigma_J^2/2}dz.$$

Noting that

$$\int_{-\infty}^{\infty} e^{-az^2+biz}dz = \frac{\sqrt{\pi}}{\sqrt{a}}\cdot e^{\frac{-b^2}{4a}}$$

(just use Maple on this for $a > 0$) gives

$$\begin{aligned} G_n(x) &= \int_{-\infty}^{\infty} e^{-(T\sigma^2/2 + n\sigma_J^2/2)z^2 + (\omega T + n\mu_J - x)iz}dz \\ &= \frac{\sqrt{2\pi}}{\sqrt{T\sigma^2 + n\sigma_J^2}}\cdot e^{-\frac{(\omega T + n\mu_J - x)^2}{2(T\sigma^2 + n\sigma_J^2)}}. \qquad (2) \end{aligned}$$

Hence the PDF is given by

$$q(x) = \frac{e^{-\lambda T}}{\sqrt{2\pi}}\left(\sum_{n=0}^{\infty}\frac{(\lambda T)^n}{n!}\cdot\frac{e^{-\frac{(\omega T + n\mu_J - x)^2}{2(T\sigma^2 + n\sigma_J^2)}}}{\sqrt{T\sigma^2 + n\sigma_J^2}}\right).$$

I also expect that one can do this in a more straightforward way using the independence of a product of normal distributions.

References

[1] A. Lewis. Fear of jumps. *Wilmott Magazine*, pages 60–67, December 2002.