Symbolic Integration of Jacobi Elliptic Functions in Maple

by

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Abstract

In this talk we will describe the design and implementation of procedures for computing indefinite integrals of Jacobi Elliptic Functions in the computer algebra system Maple. The routines take advantage of Maple's ability to extend the *int* command and are careful to return answers having a minimal size whenever possible.

Key words: Elliptic Functions, Symbolic Integration, Maple, Hermite Reduction.

1 Introduction

The original design of Maple took into consideration that many commands would need to be extensible. By this we mean that the system has a built-in knowledge of how to deal with a particular set of functions but also allows users to add additional knowledge to the system whenever possible. The best known example of this is the *diff* command for differentiating functions. One can tell Maple how to differentiate a function F by defining a procedure with name 'diff/F'. The system then knows how to handle the chain rule and all other differentiation rules that occur naturally when differentiating an expression involving F. Other examples of commands that support such extensibility include the *series* command and the *int* command [5].

In the case of the integration defining a procedure 'int/F' so that it returns say a function G also allows Maple to do some simple deductions. For example in the case of

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> int( x*F(x^2) , x)
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Maple would then be able to determine that the result is $G(x^2)/2$.

Unfortunately, in the case of integration, users have found it difficult to extend the integration knowledge base. Often this is the case because the integrands of interest involve more than one function. In this paper we will describe how we have extended Maple's int command to support the integration of the 12 Jacobian Elliptic functions along with combinations of these functions. The resulting functionality is currently included in the research version of Maple.

2 Preliminaries

The 12 Jacobi Elliptic functions $(sn(x,k), cn(x,k), dn(x,k), \ldots)$ [1] (denoted by JacobiSN(x,k), JacobiCN(x,k) etc in Maple) are doubly periodic meromorphic functions over the complex plane

[1]As in most texts we often drop the k parameter and so have $sn(x), cn(x), dn(x)$ etc.

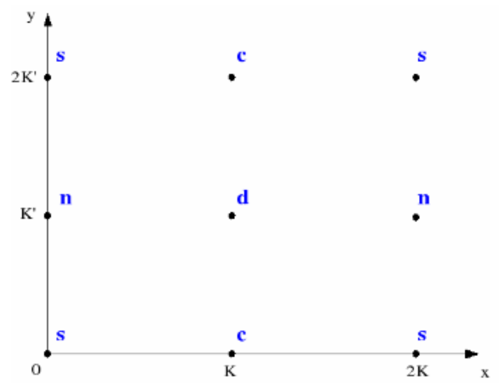

Figure 1: *Lattice describing Jacobi pattern*

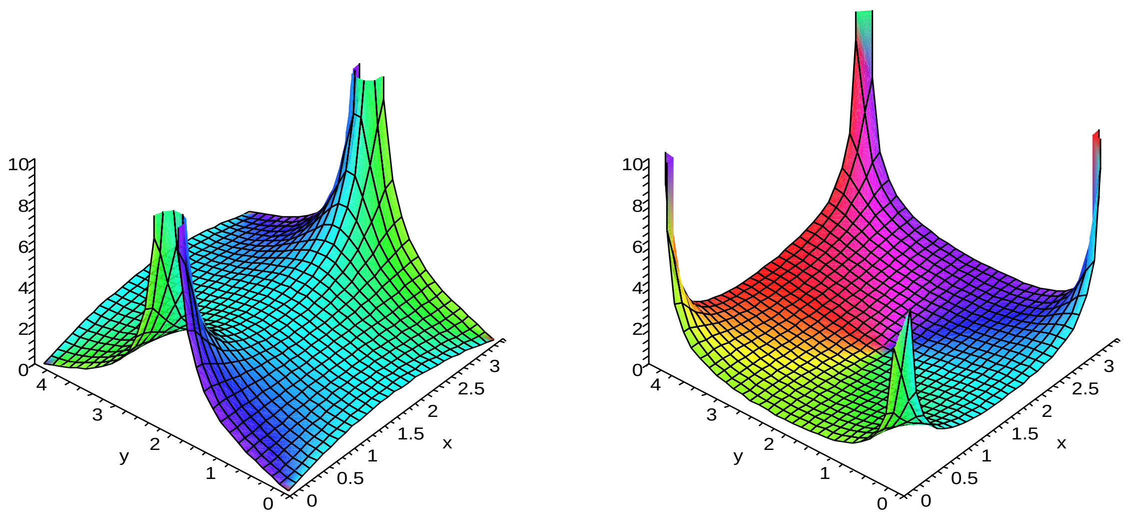

Figure 2: *The Jacobi functions* $sn(z, \frac{1}{2})$ *and* $ds(z, \frac{1}{2})$

[1]. These classical functions have been well studied since the early 19-th century and have numerous applications. The lattice diagram in Figure 1 with $K = K(k)$ and $K' = i \cdot K(\sqrt{1-k^2})$ is often used to describe the fact that the Jacobi function $pq(x, k)$ has a simple zero at p and a simple pole at q. Here $K(k) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-k^2 \sin^2\theta}}$ denotes the complete elliptic integral. Figure 2 gives a visual representation of the two functions $sn(z, \frac{1}{2})$ and $ds(z, \frac{1}{2})$ (using plots[complexplot3d] in Maple).

Formally, the Jacobi functions are defined by

$$sn(x) = \sin(\phi), \quad cn(x) = \cos(\phi), \quad dn(x) = \sqrt{1 - k^2 \sin^2 \phi}$$

where ϕ is the angle defined by

$$x = \int_0^{\phi} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}.$$

The angle ϕ is called the Jacobi amplitude (JacobiAM in Maple). The remaining 9 Jacobi functions are defined in terms of these three functions, for example $sc(x) = \frac{sn(x)}{cn(x)}$ and $ns(x) = \frac{1}{sn(x)}$. It is not surprising that there are algebraic relations between these functions which are similar to the algebraic relations between the singly periodic trigonometric functions. Indeed squares of copolar functions all have trig-style relationships. For example, we have

$$sn(x)^2 + cn(x)^2 = 1 \text{ and } k^2 sn(x)^2 + dn(x)^2 = 1.$$

A second important property of these Jacobi functions is that their derivatives are always constant multiples of the products of the copolar functions. For example we have

$$sn'(x) = cn(x)dn(x), cn'(x) = -sn(x)dn(x) \text{ and } dn'(x) = -k^2 sn(x)cn(x)$$

with similar relations for the remaining 9 functions.

3 Integration of Single Jacobi Functions

In this section we describe some of the details involved in the design and implementation of our integration procedures. In the case where we are integrating a rational expression of a single Jacobian function, the 'int/JacobiXX' mechanism is particularly effective. In all 12 cases the code first determined that one is dealing with a rational function of a single Jacobi function. The procedures then avoid unnecessary code duplication by funnelling through a single routine to compute the integral with input only the single Jacobi function and the square of its derivative. The single routine to compute the integral reduces the integral computation to an end table of six forms for each Jacobi function. We also remark that in some cases the existing literature does not provide the necessary formulas for our end tables. In these cases the formulas had to be derived using alternate sources. In fact many of these derivations made use of our tools for integrating multiple Jacobi functions.

Tables of integrals exist, for example see Byrd and Friedman [2], which provide formulas for many (but not all) of the basic mathematical information needed for mechanizing the integration of Jacobi functions. In this section we show how we make use of these formulas along with some modern technology from computer algebra (technology which actually dates from the late 1800's) for the integration of a rational function of $sn(x)$. This technology, Hermite reduction, is used to avoid unnecessary algebraic numbers or the need for polynomial factorization [3, 4]) in some cases. The same technique can be applied for any rational expression of the other 11 Jacobi functions. For any rational function R, we can always write

$$\int R(sn(x))dx = \int \frac{R(z)}{\sqrt{\omega(z)}}dz \quad \text{with } z = sn(x) \tag{1}$$

where $\omega(z) = (1-z^2)(1-k^2z^2)$. In (1) we have made use of the fact that $sn'(x)^2 = \omega(sn(x))$. A similar relation exists for the other 11 Jacobi functions and allow us to avoid code duplication.

3.1 Polynomial Integrands

From Byrd and Friedmann we obtain formulas for

$$\int sn(x)dx, \int sn(x)^2 dx \text{ and } \int sn(x)^3 dx \tag{2}$$

in terms of cn, dn, log and the elliptic integrals F and E and Π. Byrd and Friedmann also provide recurrence formulas for $\int sn(x)^k dx$ in terms of $\int sn(x)^{k-2} dx$ and $\int sn(x)^{k-4} dx$ which means that all we have a mechanical means to compute $\int P(sn(x))dx$ for all polynomials P.

In our case we prefer to make use of the following. If P is a polynomial of degree p then let $A(z)$ and $B(z)$ be polynomials with unknown coefficients of degree 3 and $p-4$, respectively. We can then set up and solve the $p+1$ linear equations in $p+1$ unknowns defined by equating coefficients in

$$P(z) = zB'(z)\omega(z) + B(z)\omega(z) + \frac{1}{2}zB(z)\omega'(z) + A(z).$$

Then one can easily verify that

$$\int P(sn(x))dx = sn(x)B(sn(x))\omega(sn(x)) + \int A(sn(x))dx.$$

Thus, using (2) along with the information that A has degree at most 3, implies that integrals of polynomial expressions of sn can be determined.

Example 3.1. Suppose $k=2$ and that we wish to compute the integral of $sn(x)^7 - 3sn(x)^5 + \frac{1031}{1024}sn(x)$. Equating the 8 coefficients on both sides of the equation

$$z^7 - 3z^5 + \frac{1031}{1024}z = (zB'(z) + B(z))(4z^4 - z^2 + 1) + \frac{1}{2}zB(z)(16z^3 - 10z) + A(z)$$

with $A(z), B(z)$ both being polynomials of degree 3 gives an 8×8 linear system. Solving this linear system gives the resulting polynomials as $A(z) = \frac{311}{384}z^3 - \frac{25}{192}z$ and $B(z) = \frac{1}{24}z^3 + \frac{25}{384}z$. This then gives

$$\begin{aligned}\int (sn(x)^7 - 3sn(x)^5 + \tfrac{1031}{1024}sn(x))dx \quad &= \quad \left(\tfrac{25}{384}\, sn\,(x) + \tfrac{1}{24}\, sn\,(x)^3\right) sn\,(x)\, cn\,(x)\, dn\,(x) \\ &\quad -\tfrac{769}{3072}\, cn\,(x)\, dn\,(x) - \tfrac{3}{16}\, sn\,(x)^2\, cn\,(x)\, dn\,(x)\,.\end{aligned}$$

As mentioned previously, this process (called the Horowitz-Ostagradsky method) can be used (with different choices of ω) for the integration of polynomials of the other 11 Jacobi functions (altering only $\omega(z)$ and the table values for the integrals of the first three powers of the respective Jacobi function).

3.2 Rational Integrands

From Byrd and Friedmann we obtain formulas for

$$\int \frac{dx}{sn(x) - a}, \quad \int \frac{dx}{1 \pm k sn(x)} \quad \text{and} \quad \int \frac{dx}{1 \pm sn(x)} \tag{3}$$

with $a^2 \neq 1$ and $a^2 \neq 1/k^2$ again in terms of cn, dn, log and the elliptic integrals F and E and Π. Therefore, when there is a square-free denominator $Q(z)$, then we can write the integral in implicit form

$$\int \frac{P(sn(x))}{Q(sn(x))}dx = \sum_{\alpha | Q(\alpha)=0} \frac{P(\alpha)}{Q'(\alpha)} \int \frac{dx}{sn(x) - \alpha}. \tag{4}$$

The above result can be represented in Maple using a sum over RootOfs construction. Of course if any of the roots of the denominator are known explicitedly then the above formulas can be refined. Similarly, even when no explicit roots are available then one can still factor the denominator over its field of coefficients and represent the answer as a sum of the expressions found in (4).

When the denominator of our rational function has repeated roots then Byrd and Friedmann also provide formulas to reduce

$$\int \frac{1}{(a - sn(x))^n} dx, \quad \int \frac{1}{(1 \pm sn(x))^n} dx \quad \text{and} \quad \int \frac{1}{(1 \pm k sn(x))^n} dx$$

in terms of lower orders. As such it is possible to obtain an answer for the integral of a rational function of $sn(x)$ simply by determining the partial fraction decomposition of the rational function and then integrating term by term.

In this section we show how modern computer algebra often eliminates the need for a complete factorization of the denominator by making use of a technique originally discovered by Hermite in the late 1800s.

Suppose now that we wish to integrate

$$\int \frac{P(sn(x))}{Q(sn(x))} dx \tag{5}$$

with P and Q relatively prime polynomials. We can assume that P has smaller degree than Q since otherwise we would simply divide the denominator into the numerator and have the polynomial remainder integrated as in the previous subsection. In addition we can compute a square-free factorization of $Q(z)$:

$$Q(z) = q_1(z) q_2(z)^2 \cdots q_n(z)^n$$

with $\gcd(q_i(z), q_j(z)) = 1$ and $\gcd(q_i(z), q_i'(z)) = 1$. Such a factorization (obtained in Maple via the command sqrfree(q,z)) can be computed via only gcd computations [4]. Using the extended Euclidean algorithm one can then decompose the integral (5) into

$$\int \frac{P(sn(x))}{Q(sn(x))} dx = \sum_{j=1}^{n} \sum_{i=0}^{j-1} \int \frac{p_{i,j}(sn(x))}{q_j(sn(x))^i} dx$$

with $p_{i,j}(z)$ having degree at most the degree of $q_j(z)$.

3.2.1 Hermite Reduction : I

Consider now the case where we are integrating

$$\int \frac{p(sn(x))}{q(sn(x))^i} dx$$

with degree $p <$ degree q and where the polynomial $q(z)$ is (1) square-free and (2) has no factors in common with $\omega(z)$. Then we can solve (via the extended Euclidean algorithm) the linear diophantine equation

$$p(z) = q(z)u(z) + q'(z)\omega(z)v(z)$$

with degree $u(z) <$ degree $q(z)$ + degree $\omega(z) - 1$ (cf. [4]). Integration by parts plus some additional algebra gives

$$\int \frac{p(sn(x))}{q(sn(x))^i} dx = \frac{v(sn(x)) sn'(x)}{(1-i) q(sn(x))^{i-1}} + \int \frac{\hat{p}(sn(x))}{q(sn(x))^{i-1}} dx \tag{6}$$

where $\hat{p}(z) = u(z) + \frac{1}{2(i-1)}(2v'(z)\omega(z) + v(z)\omega'(z))$. This gives a reduction procedure resulting in

$$\int \frac{p(sn(x))}{q(sn(x))^i} dx = \frac{p_1(sn(x))}{q(sn(x))^{i-1}} + \cdots + \frac{p_{i-2}(sn(x))}{q(sn(x))} + \int \frac{p_{i-1}(sn(x))}{q(sn(x))} dx \tag{7}$$

with $p_1(z), \ldots, p_{i-1}(z)$ polynomials. Note that it may happen that the degree of $p_{i-1}(z)$ is greater than or equal to the degree of $q(z)$. Polynomial division combined with the integration of polynomials of $sn(x)$ then leaves the remaining problem being one where the numerator has degree less than the denominator.

Example 3.2. Let $k = 2$. Then Hermite reduction for the integrand $\frac{411 sn(x)^2 - 1230 sn(x) - 5544}{(sn(x)^3+2)^2}$ results in

$$\begin{aligned} \int \frac{411 sn(x)^2 - 1230 sn(x) - 5544}{(sn(x)^3+2)^2} dx &= \frac{\left(50\, sn(x)^2 - 160\, sn(x) + 389\right) cn(x) dn(x)}{3\left(sn(x)^3+2\right)} \\ & 502\, x + \frac{50}{3}\, E\left(sn\left(x\right), 2\right) - \int \frac{3756}{sn(x)^3+2} dx. \end{aligned}$$

3.2.2 Hermite Reduction : II

Suppose now that $q(z)$ is a factor of $\omega(z)$ with $\omega(z) = q(z) \cdot c(z)$ so that $q(z)$ and $\omega(z)$ are no longer relatively prime. Then one can solve the linear diophantine equation

$$p(z) = q(z)u(z) + q'(z)c(z)v(z)$$

in order to obtain the reduction formula

$$\int \frac{p(sn(x))}{q(sn(x))^i} dx = \frac{2v(sn(x))}{(1-2i)q(sn(x))^i} + \int \frac{\hat{p}(sn(x))}{q(sn(x))^{i-1}} dx \tag{8}$$

with $\hat{p}(z) = u(z) + \frac{1}{2i-1}(2v'(z)c(z) + v(z)c'(z))$. Therefore

$$\int \frac{p(sn(x))}{q(sn(x))^i} dx = \frac{p_1(sn(x))}{q(sn(x))^i} + \cdots + \frac{p_{i-2}(sn(x))}{q(sn(x))^2} + \int \frac{p_{i-1}(sn(x))}{q(sn(x))} dx \tag{9}$$

with $p_1(z), \ldots, p_{i-1}(z)$ polynomials. Again it might happen that the degree of $p_{i-1}(z)$ is not less than the degree of $q(z)$.

Note that if $a(z) = b(z)c(z)$ with $\gcd(b(z), c(z)) = 1$ then

$$\frac{p(z)}{a(z)^i} = \frac{u(z)}{b(z)^i} + \frac{v(z)}{c(z)^i}$$

where $u(z), v(z)$ solve the linear diophantine equation

$$p(z) = u(z)c(z)^i + v(z)b(z)^i.$$

As such one can always reduce an integral of the form $\int p(sn(x))/q(sn(x))^i dx$ to one of the above two Hermite reduction cases.

Example 3.3. In the case of integrand $\frac{sn(x)^2 - (1-k^2) sn(x)^4 + k^2 sn(x)^6}{(sn(x)^2+1)^3}$ we can determine the complete internal via

$$\begin{aligned} \int \frac{k^2 sn(x)^6 - (1-k^2) sn(x)^4 + sn(x)^2}{(sn(x)^2+1)^3} dx &= \frac{3k^2}{4} x - \frac{sn(x) cn(x) dn(x)}{4(sn(x)^2+1)^2} + \frac{\left(3+5k^2\right) sn(x) cn(x) dn(x)}{16(1+k^2)(sn(x)^2+1)} - \frac{k^2\left(3+5k^2\right) x}{16(1+k^2)} \\ & - \frac{\left(3+5k^2\right)(x - E(sn(x),k))}{16(1+k^2)} - \frac{\left(-1+10k^2+7k^4\right) \Pi(sn(x),-1,k)}{16(1+k^2)}. \end{aligned}$$

Notice that the roots $\pm i$ of the denominator z^2+1 do not appear in the final result nor is there any need for arithmetic in the rationals extended by i in this case.

4 Integration with Multiple Jacobi Functions

In the case where there are multiple Jacobi functions in the integrand, there are reduction formulas which allow one to break down these cases to groups which eventually reduce to the case of one integration of a rational functions of a single Jacobi function or to the integration of a rational function. In this case one of the issues includes deciding which one of many reductions should be used to obtain the smallest and most useful output in most cases.

Assume first that the integrand consists of a rational function of $sn(x)$ and its two copolar functions $cn(x)$ and $dn(x)$. Then, as mentioned in Byrd and Friedmann, the trig-style identities between these functions implies that

$$R(sn(x), cn(x), dn(x)) = R_1(sn(x)) + R_2(sn(x))cn(x) + R_3(sn(x))dn(x) + R_4(sn(x))cn(x)dn(x)$$

where R_1, R_2, R_3 and R_4 are all rational functions. Continuing further, Byrd and Friedmann point out that

$$\begin{aligned}
\int R_2(sn(x))cn(x)dx &= 2\int \tfrac{1}{1+k^2t^2} R_2(\tfrac{2t}{1+k^2t^2})dt \quad && \text{with } t = \tfrac{sn(x)}{dn(x)+1} \\
\int R_3(sn(x))dn(x)dx &= 2\int \tfrac{1}{1+t^2} R_3(\tfrac{2t}{1+t^2})dt && \text{with } t = \tfrac{sn(x)}{cn(x)+1} \\
\int R_4(sn(x))cn(x)dn(x)dx &= \int R_4(t)dt && \text{with } t = sn(x)
\end{aligned}$$

while of course $\int R_1(sn(x))dx$ is determined using the methods of the previous sections.

In addition there are instances where we take advantage of much simpler changes of variable. In particular, we have the formulas

$$\int R(cn(x))sn(x)dn(x)dx = -\int R(t)dt \text{ with } t = cn(x)$$

and

$$\int R(dn(x))sn(x)cn(x)dx = -\frac{1}{k^2}\int R(t)dt \text{ with } t = dn(x).$$

Thus for example, integrals of the form

$$\begin{aligned}
\int sn(x)^m cn(x)^\ell dn(x)^n dx &= \int p_1(t)dt \quad && \text{with } t = sn(x) \text{ when } \ell, n \text{ odd} \\
&= \int p_2(t)dt && \text{with } t = cn(x) \text{ when } m, n \text{ odd} \\
&= \int p_3(t)dt && \text{with } t = dn(x) \text{ when } m, \ell \text{ odd}
\end{aligned}$$

and so forth (where p_1, p_2, p_3 are all polynomials).

There are similar reductions for all other rational functions of a single Jacobi function along with its two copolar functions. We have implemented this using the same code for the groups $\{sc(x), nc(x), dc(x)\}$, $\{sd(x), nd(x), cd(x)\}$ and $\{cs(x), ns(x), ds(x)\}$. In all other cases we convert all input functions into rational functions of sn, cn and dn.

Interestingly enough one can use the above formulas to obtain those formulas that are needed for our integrator but which do not appear in Byrd and Friedmann. For example, the formula

$$\int \frac{dx}{sd(x)-\alpha} = -\frac{\alpha^2 k^2 x + \Pi\left(sn(x), \frac{1+\alpha^2k^2}{\alpha^2}, k\right)}{1+\alpha^2k^2} + \frac{arctanh\left(\frac{\left(\frac{\alpha^2(sd(x))^2}{2(cd(x)+nd(x))^2} - 1 + \frac{\alpha^2}{2} - \alpha^2k^2\right)}{\sqrt{1-\alpha^2+2\alpha^2k^2-\alpha^4k^2+\alpha^4k^4}}\right)}{\sqrt{1-\alpha^2+2\alpha^2k^2-\alpha^4k^2+\alpha^4k^4}}$$

needed for integrating rational functions of $sc(x)$ was determined by conversion to

$$\int \frac{dn(x)}{sn(x)-\alpha dn(x)}dx = \int \frac{sn(x)}{(1+\alpha^2k^2)sn(x)^2-\alpha^2} dn(x)dx - \alpha\int \frac{k^2sn(x)^2-1}{(1+\alpha^2k^2)sn(x)^2-\alpha^2}dx.$$

5 Conclusion

In this paper we have given some of the details about the mathematical algorithms and design decisions that were involved in incorporating a procedure for integrating Jacobi elliptic functions in Maple. We have described the case of a rational function in $sn(x)$. However the same code is used for rational functions of any of the 12 Jacobi functions. In the case of rational expressions containing multiple Jacobi functions we follow a process of reduction into four or less integrals of rational functions. In the case where multiple answers are possible an important consideration in our work was returning closed form solutions that were as simple as possible. Of course all results found in Byrd and Friedmann [2], the standard reference, are produced by our code (a minimum goal) and verified through differentiation and simplification. In some cases incorrect answers were found in [2] and corrected (as one could easily guess, all errors were minor typos).

For future research we mention that we still wish to find the minimum form of any closed form solution returned by our code. In the case of a rational expression of a single Jacobian function this would ensure that a minimum number of algebraic numbers are used to represent a given Jacobi elliptic integral. This is the case for integration of rational functions and also for integration of algebraic functions when the result is elementary (cf. [3, 4]). Finally, it remains to consider the case of definite integration where changes of variables may result in branching issues.

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