Q3. A is $n \times n$, nonsingular.
F and G are $n \times p$, where $1 \le p << n$.

We are to compute $\underbrace{W}_{n\times n} = \underbrace{F}_{n\times p}\underbrace{G^T}_{p\times n}\underbrace{A^{-1}}_{n\times n}$.

(a) (left-to-right evaluation)

Step 1: $\underbrace{V}_{n\times n} = \underbrace{F}_{n\times p}\underbrace{G^T}_{p\times n}$.

For each of the n^2 entries in V we must compute
(row i of F) $\cdot$ (col. j of G^T) ← dot product
or (row i of F) $\cdot$ (row j of G)

which is (vector length p) $\cdot$ (vector length p)
$\Longrightarrow 2p + O(1)$ flops.

Total cost for Step 1 = $\boxed{2pn^2 + O(n^2) \text{ flops}}$

Step 2: $W = VA^{-1} \Longrightarrow WA = V \Rightarrow \underbrace{A^T}_{n\times n}\underbrace{W^T}_{n\times n} = \underbrace{V^T}_{n\times n}$
implies that W can be computed via Matlab command
$W = (A' \backslash V')'$ (' denotes transpose).

For a single vector $\underline{b}$ of length n, $A \backslash \underline{b}$
costs $\frac{2}{3}n^3 + O(n^2)$ flops. [It does LU factorization followed by solving.]
But we have A' and V' both matrices of size $n \times n$
so the above command to compute W costs as follows.

We can view it as n linear system problems to solve, each with the same coefficient matrix A'.

So,

Cost for Step 2 = 1 LU factorization of A' + $n *$ (Forward + Back Subst.)

$$= \frac{2}{3}n^3 + O(n^2) \text{ flops}$$

$$+ n\,(2n^2 + O(n)) \text{ flops}$$

$$= \boxed{\frac{8}{3}n^3 + O(n^2) \text{ flops}}$$

Finally, adding the cost of Step 1 and Step 2,

Total cost for method (a) = $\boxed{\frac{8}{3}n^3 + 2pn^2 + O(n^2) \text{ flops}}$

Q3(b). A better method arises by noting the sizes of the matrices involved (see top of previous page).

In method (a), we formed $\underbrace{F}_{n\times p}\ \underbrace{G^T}_{(p\times n)}$ yielding a matrix V of size $n\times n$. But p is assumed to be much smaller than n so it would be less costly to execute the evaluation from right-to-left because:

$$\underbrace{X}_{p\times n} = \underbrace{G^T}_{p\times n}\ \underbrace{A^{-1}}_{n\times n} \quad \text{is of size } p\times n \text{ (not } n\times n\text{).}$$

Right-to-left evaluation

Step 1: $X = G^T A^{-1} \implies XA = G^T$

$$\implies \underbrace{A^T}_{n\times n}\ \underbrace{X^T}_{n\times p} = \underbrace{G}_{n\times p}$$

I.e., $X = (A' \backslash G)'$.

Since G is $n\times p$, we have only p linear system problems to solve.

Cost for Step 1 = 1 LU factorization of A' + $p *$ (Forward + Back Subst.)

$$= \left[\tfrac{2}{3}n^3 + O(n^2)\right] + \left[p(2n^2 + O(n))\right]$$

$$= \boxed{\tfrac{2}{3}n^3 + 2pn^2 + O(n^2) \text{ flops}}$$

Step 2: $\underbrace{W}_{n \times n} = \underbrace{F}_{n \times p} \underbrace{X}_{p \times n}$.

For each of the n^2 entries in W we must compute

(row i of F) $\cdot$ (col j of X)

$\Rightarrow$ (vector length p) $\cdot$ (vector length p)

$\Rightarrow$ $2p + O(1)$ flops.

Total cost for Step 2 = $\boxed{2pn^2 + O(n^2) \text{ flops}}$

Finally, adding the cost of Step 1 and Step 2,

Total cost for method (b) =

$$\boxed{\frac{2}{3}n^3 + 4pn^2 + O(n^2) \text{ flops}}.$$