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CS 370 Fall 2008: Assignment 2 SOLUTIONS

Question 2

The cubic spline $S(x)$ for this problem consists of two cubic polynomials which are given in the following form:

$$\begin{aligned} p_1(x) &= a + b\,(x-1) + c\,(x-1)^2 - \tfrac{1}{4}\,(x-1)^2\,(x-2)\,, \quad 1 \le x \le 2 \\ p_2(x) &= e + f\,(x-2) + g\,(x-2)^2 + \tfrac{1}{4}\,(x-2)^2\,(x-3)\,, \quad 2 \le x \le 3\,. \end{aligned}$$

In the various parts of this question we proceed to verify the conditions which ensure that this is indeed a natural cubic spline, and we determine values for the unknown coefficients.

(a) We are given that $S(x)$ must interpolate the data (1, 1), (2, 1), (3, 0) which leads to the following 4 equations:

$$p_1(1) = 1 \implies a = 1 \qquad (1)$$

$$p_1(2) = 1 \implies a + b + c = 1 \qquad (2)$$

$$p_2(2) = 1 \implies e = 1 \qquad (3)$$

$$p_2(3) = 0 \implies e + f + g = 0\,. \qquad (4)$$

Therefore, we already know the values for two of the coefficients:

$$a = 1; \;\; e = 1\,.$$

(b) The *1st derivative condition* at the interior point $x = 2$ yields the following equation:

$$p_1'(2) = p_2'(2) \implies b + 2\,c - \frac{1}{4} = f\,. \qquad (5)$$

(c) The *natural boundary conditions* at the two end points yield the following 2 equations:

$$p_1''(1) = 0 \implies 2\,c + \frac{1}{2} = 0 \qquad (6)$$

$$p_2''(3) = 0 \implies 2\,g + 1 = 0 \qquad (7)$$

which are trivially solved to give

$$c = -\frac{1}{4}; \;\; g = -\frac{1}{2}\,.$$

(d) Plugging the values already determined for a, e, c and g into the equations in part (a), we are left with just two equations (from equations (2) and (4)):

$$1 + b - \frac{1}{4} = 1$$

$$1 + f - \frac{1}{2} = 0$$

which are trivially solved to give

$$b = \frac{1}{4}; \;\; f = -\frac{1}{2}\,.$$

(e) Considering all the conditions that define a natural cubic spline, what condition has not yet been checked? It is the *2nd derivative condition* at the interior point $x = 2$ that has not been taken into account, namely:

$$p_1''(2) = p_2''(2)\,. \tag{8}$$

It can be verified that $p_1''(2) = p_2''(2) = -\frac{3}{2}$ so the continuity of $S''(x)$ is assured.
It may also be noted that the continuity of the 1st derivative, $S'(x)$, was handled in part (b) which yielded equation (5) but we haven't actually checked that this equation holds. For the values of b, c and f determined in parts (c) and (d), one can verify that equation (5) does indeed hold.