An ϵ-monotone Fourier method for Guaranteed Minimum Withdrawal Benefit as a continuous impulse control problem *

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Abstract

When formulated as an impulse control problem, the no-arbitrage pricing of Guaranteed Minimum Withdrawal Benefit contracts with continuous withdrawals results in a Hamilton-Jacobi-Bellman Quasi-Variational Inequality (HJB-QVI), which must be solved numerically. In this paper, using an associated Green's function, we develop a numerical Fourier method which is only monotone within a tolerance $\epsilon > 0$ to solve the above HJB-QVI under jump-diffusion dynamics. We appeal to a Barles-Souganidis-type analysis in [14], which is originally developed for strictly monotone scheme, to rigorously prove the convergence of our scheme to the viscosity solution of the HJB-QVI as $\epsilon \to 0$. Extensive numerical experiments demonstrate an excellent agreement with benchmark results obtained by finite difference methods and Monte Carlo simulation.

Keywords: Variable annuities, guaranteed minimum withdrawal benefit, impulse control, HJB equation, Fourier series, viscosity solution, monotonicity

AMS Classification 65N06, 93C20

1 Introduction

In a continuous withdrawal setting, the no-arbitrage pricing problem of Guaranteed Minimum Withdrawal Benefit (GMWB) contracts can be formulated using either impulse control or singular control, typically resulting in an Hamilton-Jacobi-Bellman Quasi-Variational Inequality (HJB-QVI). This HJB-QVI must be solved numerically, since a closed-form solution for it is not known to exist. The reader is referred to [15, 24, 40, 41, 42, 54] and [7, 19, 20] for an analysis of singular control and impulse control formulations, respectively. Generally speaking, the impulse control approach is suitable for many complex situations in stochastic optimal control [3, 8, 16, 25, 31, 37, 46, 57, 64]. For GMWB contracts, impulse control is more convenient than singular control in handling complex contract features, such as is the reset provision[1, 24, 26, 38, 54, 67].

Provable convergence of numerical methods for HJB equations are typically built upon the framework established by Barles and Souganidis in [14]. This framework requires numerical methods to be (i) monotone (in the viscosity sense), (ii) stable, and (iii) consistent. Among these requirements, monotonicity is often the most challenging to achieve, and consistency in the viscosity sense is usually the most difficult to prove theoretically, especially for equations with complex boundary conditions. Non-monotone schemes could produce numerical solutions that fail to converge to viscosity solutions, resulting in a

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violation of the no-arbitrage principle [55, 59, 68]. When a finite difference method is used, monotonicity is ensured by a positive coefficient discretization method [34, 52, 59, 66].[1] In the context of pricing GMWB contracts with continuous withdrawal, convergence of finite difference scheme to the viscosity solution of the associated HJB-QVI is studied in great detail in [19, 20, 24, 40, 41, 42].

Pricing GMWB contracts with discrete withdrawals typically involves solving, between fixed intervention times, either (i) an associated linear Partial-Integro Differential Equation (PIDE) using finite differences [19, 24], or (ii) an expectation problem using numerical integration [1, 15, 44, 45, 51, 62], or regression-type Monte Carlo [9, 43]. Across intervention times, an optimization problem needs to be solved. Numerical integration Fourier-based methods often depend on the availability of a closed-form expression of the Fourier transform of the underlying transition density function or an associated Green's function [1, 45]. It is well-known that, if applicable, Fourier-based methods offer several important advantages over finite differences, such as no timestepping error between intervention times, and the capability of straightforward handling of realistic underlying dynamics, such as jump diffusion and regime-switching. However, a major drawback of existing Fourier-based methods is their lack of strict monotonicity. This issue is particularly problematic in the context of stochastic optimal control in general where optimal decisions are determined by comparing numerically computed value functions. Furthermore, another challenge with Fourier-based methods is potential wraparound contamination in numerical solutions. This matter is also crucial to stochastic optimal control problems, especially to impulse control formulations, due to the non-local nature of impulses. To the best of our knowledge, both of these deficiencies of Fourier-based methods have not been addressed adequately in the impulse control literature. The reader is referred to [18, 23, 33, 49, 50] for analysis of error bounds, and [1, 45] for zero padding techniques in GMWB pricing.

The main focus of this paper is the development of a provably convergent Fourier method to tackle the challenging HJB-QVI arising from an impulse control formulation of GMWB contracts under jump-diffusion dynamics. Major contributions of the paper are as follows.

- We propose the pricing problem of GMWB contracts with continuous withdrawals under jump-diffusion dynamics [47, 53] as an HJB-QVI posed on an infinite definition domain consisting of a finite interior and infinite boundary sub-domains with appropriate boundary conditions.
- Using the Green's function of an associated PIDE, we study proper truncation of boundary sub-domains to ensure loss of information is negligible. We then develop a Fourier scheme which is monotone within a tolerance $\epsilon > 0$ to solve the above HJB-QVI on a finite computational domain. Under a suitable growth condition, the scheme is shown to be ℓ_∞-stable and consistent in the viscosity sense with respect to the HJB-QVI defined on the infinite domain.
- We propose an efficient implementation of the scheme via Fast Fourier Transform, including a proper handling of boundary conditions and padding techniques. We mathematically prove that our padding techniques, whilst simple, can effectively control wraparound errors in the numerical solutions.
- We prove a strong comparison principle result for the finite interior sub-domain and parts of its boundary. We then appeal to a Barles-Souganidis-type analysis in [14], to rigorously prove the convergence of our scheme the unique viscosity solution of the HJB-QVI as the discretization parameter and the monotonicity tolerance ϵ approach zero.
- Numerical experiments confirm excellent agreement with benchmark results obtained by finite difference methods and Monte Carlo simulation, as well as the robustness of the proposed ϵ-monotone Fourier method. Through experiments, we also numerically show that inadequate treatments of

[1] When dealing with cross derivative terms, a wide-stencil method based on a local coordinate rotation can be used to construct monotone finite difference schemes [28, 52, 52]; however, this could be computationally expensive.

padding areas could significantly contaminate the numerical solutions of the impulse control formulation.

Although we focus specifically on GMWB, our comprehensive and systematic approach could serve as a numerical and convergence analysis framework for the development of similar weakly monotone methods for HJB-QVIs arising from impulse control problems in finance.

2 Underlying processes

This section briefly reviews the impulse control formulation [7, 19, 20] and introduces the notation to be used in this paper. We respectively denote by $Z(t)$ and $A(t)$ the balance of the personal sub-account and of the guarantee account at time t, $t \in [0, T]$, where $T > 0$ is a fixed investment horizon. Let z_0 be the up-front premium to the insurer, which is placed in the personal account at the inception of the contract, hence $Z(0) = z_0$. The policy guarantees that the sum of withdrawals throughout the policy's life is equal to the premium, hence $A(0) = z_0$. For subsequent use, let $t^- = t - \varepsilon$, where $\varepsilon \downarrow 0^+$.

Roughly speaking, the holder of the policy can either (i) withdraw continuously at a rate determined by the holder, or (ii) withdraw specific amounts at specific times, both determined by the holder, subject to a penalty charge imposed by the insurer. Regarding (i), as almost all policies with a GMWB have a cap on the maximum allowed continuous withdrawal rate without penalty [24], we let C_r be such a contractual (continuous) withdrawal rate. For (ii), withdrawing a finite amount is subject to a penalty charge proportional to the withdrawal amount as well as a strictly positive fixed cost. We let $\mu < 1$ be the positive penalty rate, and c be the positive fixed cost.

Under an impulse control framework [46, 57], the time-t control for the holder consists of (i) a continuous control $\hat{\gamma}(t)$, $\hat{\gamma}(t) \in [0, C_r]$, representing continuous withdrawal rate, and (ii) an impulse control $\{(t^k, \gamma^k)\}_{k \le K}$, $K \le \infty$, representing intervention times $\{t^k\}_{k \le K}$ and associated impulses $\{\gamma^k\}_{k \le K}$, where each t^k corresponds to a time at which the holder instantaneously withdraws a finite amount, and γ^k, $\gamma^k \in [0, A(t^{k-})]$, corresponds to the withdrawal amount at that time. Here, $\{t^k\}_{k \le K}$ is any sequence of $(\mathcal{F}_t)$-stopping times satisfying $0 \le t \le t^1 \le t^2 < \ldots < t^K \le T$, and $\{\gamma^k\}_{k \le K}$ is a corresponding sequence of random variables with each γ^k being of $\mathcal{F}_{t^k}$-measurable for all t^k. Due to penalty charge, the net revenue cash flow provided to the policy holder at time t^k is $(1 - \mu)\gamma^k - c$.

As shown in [24], the dynamics of $A(t)$ are given by

$$
\begin{aligned}
dA(t) &= -\hat{\gamma}(t)\mathbf{1}_{\{A(t)>0\}}dt, \quad \text{for} \quad t \neq t^k, \quad k = 1, 2, \ldots, K, \\
A(t) &= A(t^-) - \gamma^k, \qquad \text{for} \quad t = t^k, \quad k = 1, 2, \ldots, K.
\end{aligned} \tag{2.1}
$$

The dynamics of $Z(t)$ are assumed to follow

$$
\begin{aligned}
\frac{dZ(t)}{Z(t)} &= (r - \beta - \lambda\kappa)\, dt + \sigma dW(t) + d\left(\sum_{i=1}^{\pi(t)} (\psi_i - 1)\right) - \hat{\gamma}(t)\mathbf{1}_{\{Z(t),A(t)>0\}}dt, \\
& \qquad\qquad\qquad\qquad\qquad \text{for} \quad t \neq t^k, \quad k = 1, 2, \ldots, K, \\
Z(t) &= \max\Big(Z(t^-) - \gamma^k, 0\Big), \qquad \text{for} \quad t = t^k, \quad k = 1, 2, \ldots, K.
\end{aligned} \tag{2.2}
$$

In (2.2), $W(t)$ denotes a standard Wiener process, $r > 0$ and $\sigma > 0$ are the risk-free rate and volatility, respectively, β is the proportional annual insurance rate paid by the policy holder, and $\pi(t)$ is a Poisson process with intensity $\lambda \ge 0$. Denote by ψ the random number representing the jump multiplier, and $\kappa = \mathbb{E}[\psi - 1]$ with $\mathbb{E}[\cdot]$ being the expectation operator. Finally, ψ_i are i.i.d. random variables having the same distribution as ψ with ψ_i, $\pi(t)$ and $W(t)$ assumed to all be mutually independent. The mean and variance of ψ are assumed to be finite.

As a specific example for dynamics (2.2), we consider two jump distributions for ψ, namely the log-normal distribution [53] and the log-double-exponential distribution [47]. Let $b(y)$ be the density of $\ln \psi$. In the first case, $\ln \psi$ is normally distributed with mean ν and standard deviation ς, with $b(y)$ given by

$$
b(y) = \frac{1}{\varsigma\sqrt{2\pi}} \exp\left\{-\frac{(y - \nu)^2}{2\varsigma^2}\right\}. \tag{2.3}
$$

In the latter case, $\ln\psi$ has an asymmetric double-exponential distribution, with $b(y)$ given by

$$b\,(y) = p_u \eta_1 e^{-\eta_1 y}\mathbf{1}_{\{y\geq 0\}} + (1-p_u)\,\eta_2 e^{\eta_2 y}\mathbf{1}_{\{y<0\}}. \tag{2.4}$$

Here, $p_u \in [0,1]$, $\eta_1 > 1$ and $\eta_2 > 0$. Given that a jump occurs, p_u is the probability of an upward jump, and $(1-p_u)$ is the probability of a downward jump.

3 Impulse control formulation

For the controlled processes $(Z(t), A(t))$, $t \in [0,T]$, let (z,a) be the state of the system. We denote by $u(z,a,t)$ the time-t no-arbitrage price of a variable annuity with a GMWB when $Z(t) = z$ and $A(t) = a$. Using dynamic programming, $u(z,a,t)$ is shown to satisfy the impulse control formulation [4, 19]

$$\min\Bigg\{ -u_t - \mathcal{L}'u - \mathcal{J}'u - \sup_{\hat{\gamma}\in[0,C_r]} \hat{\gamma}\left(1 - u_z\mathbf{1}_{\{z>0\}} - u_a\right)\mathbf{1}_{\{a>0\}},$$

$$u - \sup_{\gamma\in[0,a]}\left[u\left(\max\left(z-\gamma,0\right), a-\gamma, t\right) + (1-\mu)\,\gamma - c\right]\Bigg\} = 0, \tag{3.1}$$

where $(z,a,t) \in [0,\infty)\times[a_{\min}, a_{\max}]\times[0,T)$. Here, $a_{\min} = 0$ and $a_{\max} = z_0$ and

$$\mathcal{L}'u\,(z,a,t) = \frac{\sigma^2 z^2}{2}u_{zz} + (r - \lambda\kappa - \beta)\,z u_z - (r+\lambda)\,u, \quad \mathcal{J}'u\,(z,a,t) = \lambda\int_{-\infty}^{\infty} u\left(ze^{y}, a, \tau\right) b(y)\;dy, \tag{3.2}$$

with $b(\cdot)$ being the probability density function of $\ln\psi$. We note that the fixed cost c is introduced as a technical tool to ensure uniqueness of the impulse formulation, as commonly done in the impulse control literature [57, 58, 65].

Let $\tau = T - t$, and for $z > 0$, we apply the change of variable $w = \ln(z) \in (-\infty,\infty)$. Let $\mathbf{x} = (w, a, \tau)$, and denote by $v(\mathbf{x}) \equiv v(w,a,\tau) = u(e^w, a, T-t)$. Since $\log(\cdot)$ is undefined at zero, in (3.1), under the log-transformation in z, the term $\max(u-\gamma, 0)$ becomes $\ln\left(\max\left(e^w - \gamma, e^{w_{-\infty}}\right)\right)$ for a finite $w_{-\infty} \ll 0$. With these in mind, formulation (3.1) becomes

$$\min\Bigg\{ v_\tau - \mathcal{L}v - \mathcal{J}v - \sup_{\hat{\gamma}\in[0,C_r]} \hat{\gamma}\left(1 - e^{-w}v_w - v_a\right)\mathbf{1}_{\{a>0\}},$$

$$v - \sup_{\gamma\in[0,a]}\left[v\left(\ln\left(\max\left(e^w - \gamma, e^{w_{-\infty}}\right)\right), a-\gamma, \tau\right) + (1-\mu)\,\gamma - c\right]\Bigg\} \;=\; 0, \tag{3.3}$$

where $(w,a,\tau) \in \Omega^{\infty} \;\equiv\; (-\infty,\infty)\times[a_{\min},a_{\max}]\times[0,T)$, and $\mathcal{L}(\cdot)$ and $\mathcal{J}(\cdot)$ are defined by

$$\mathcal{L}v\,(\mathbf{x}) = \frac{\sigma^2}{2}v_{ww} + (r - \frac{\sigma^2}{2} - \lambda\kappa - \beta)v_w - (r+\lambda)v, \quad \mathcal{J}v\,(\mathbf{x}) = \lambda\int_{-\infty}^{\infty} v(w+y, a, \tau)\;b(y)\;dy. \tag{3.4}$$

3.1 Localization

Under the log transformation, the GBMW formulation (3.3) is posed on the infinite domain Ω^{∞}. For the problem statement and convergence analysis of numerical schemes, we define a localized GMWB impulse formulation. To this end, with $w_{\min} < 0 < w_{\max}$, $|w_{\min}|$ and $w_{\max}$ sufficiently large, we define the following sub-domains:

$$\begin{aligned}
\Omega^{\infty}_{\tau_0} &= (-\infty,\infty)\times[a_{\min},a_{\max}]\times\{0\},\\
\Omega^{\infty}_{w_{\max}} &= [w_{\max},\infty)\times[a_{\min},a_{\max}]\times(0,T],\\
\Omega^{\infty}_{w_{\min}} &= (-\infty, w_{\min}]\times(a_{\min},a_{\max}]\times(0,T],\\
\Omega_{a_{\min}} &= (w_{\min}, w_{\max})\times\{a_{\min}\}\times(0,T],\\
\Omega^{\infty}_{wa_{\min}} &= (-\infty, w_{\min}]\times\{a_{\min}\}\times(0,T],\\
\Omega_{\text{in}} &= \Omega^{\infty}\setminus\Omega^{\infty}_{\tau_0}\setminus\Omega^{\infty}_{w_{\min}}\setminus\Omega^{\infty}_{wa_{\min}}\setminus\Omega^{\infty}_{w_{\max}}\setminus\Omega_{a_{\min}},\\
\partial\Omega_{\text{in}} &= \Omega_{a_{\min}}\cup(w_{\min},w_{\max})\times[a_{\min},a_{\max}]\times\{0\}\\
&\quad\cup\{w_{\min},w_{\max}\}\times[a_{\min},a_{\max}]\times[0,T].
\end{aligned} \tag{3.5}$$

An illustration of the sub-domains for the localized problem is given in Figure 3.1.

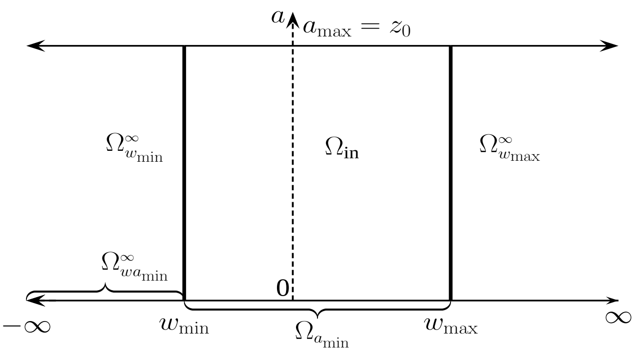

FIGURE 3.1: *Spatial computational domain at each τ.*

We now present equations for sub-domains defined in (3.5). We note that boundary conditions for $\tau \to 0$, $w \to -\infty$, $w \to \infty$, and $a \to a_{\min}$ are obtained by relevant asymptotic forms of the HJB-QVI (3.1) when $t \to T$, $z \to 0$, $z \to \infty$, and $a \to a_{\min}$, respectively, similar to [19, 24]. We also note that the initial and boundary solutions in $\Omega^{\infty}_{\tau 0}$ and $\Omega^{\infty}_{w_{\max}}$ may grow unbounded as $w \to \infty$. Therefore, to ensure boundedness of numerical solutions in the interior sub-domains $\Omega_{\text{in}} \cup \Omega_{a_{\min}}$, where convergence to the unique viscosity solution is studied, we require the initial and boundary solutions in $\Omega^{\infty}_{\tau 0}$ and $\Omega^{\infty}_{w_{\max}}$ to be bounded as $w \to \infty$. This is detailed below.

- For $(w, a, \tau) \in \Omega_{\text{in}}$, we have (3.3).

- For $(w, a, \tau) \in \Omega^{\infty}_{\tau 0}$, we use the initial condition $v(w, a, 0) = \max(e^w, (1-\mu)a - c) \wedge e^{w_\infty}$ for a finite $w_\infty \gg w_{\max}$, where $x \wedge y = \min(x, y)$.

- For $(w, a, \tau) \in \Omega^{\infty}_{w_{\max}}$, according to [24], the withdrawal guarantee becomes insignificant for w sufficiently large. As suggested in [40], the exact boundary condition at point $(w, a, \tau) \in \Omega^{\infty}_{w_{\max}}$ is $v(w, a, \tau) = e^{-\beta\tau} e^w \left(1 + \mathcal{O}\left(\frac{a_{\max}}{e^w}\right)\right)$. Therefore, following [24, 40], in $\Omega^{\infty}_{w_{\max}}$, we impose the (bounded) Dirichlet-type boundary condition

$$v = e^{-\beta\tau}(e^w \wedge e^{w_\infty}). \tag{3.6}$$

  We note that the theoretical quantity w_∞ is needed to indicate that the solutions $\Omega^{\infty}_{\tau 0}$ and $\Omega^{\infty}_{w_{\max}}$ are bounded as $w \to \infty$, and it does not need to be numerically specified. It is possible to relax this boundedness requirement to an exponential growth via a simple change of variable (see, for example, [32][Remark 3.7]).

- As $w \to -\infty$, $z = e^w \to 0$. Set $z = 0$ in (3.1), and then transform back to the $w = \ln z$ coordinates to obtain

$$\min\left\{ v_\tau + rv - \sup_{\hat{\gamma}\in[0,C_r]} \hat{\gamma}\,(1 - v_a)\,\mathbf{1}_{\{a>0\}},\ v - \sup_{\gamma\in[0,a]} [v(w, a-\gamma, \tau) + \gamma(1-\mu) - c] \right\} = 0. \tag{3.7}$$

  Further justification of this boundary condition is given in [24]. We use the boundary condition (3.7) for point $(w, a, \tau) \in \Omega^{\infty}_{w_{\min}}$. This is essentially a Dirichlet boundary condition since it can be solved independently without using any information other than from $\Omega^{\infty}_{w_{\min}}$.

- For $(w, a, \tau) \in \Omega_{a_{\min}}$, the impulse formulation becomes the linear PDE $v_\tau - \mathcal{L}v - \mathcal{J}v = 0$ which can be solved independently without using any information other than at $a = 0$.

- For $(w, a, \tau) \in \Omega^{\infty}_{wa_{\min}}$, (3.7) becomes $v_\tau + rv = 0$.[2]

Note that no further information is needed along the boundary $a = a_{\max}$ due to the hyperbolic nature of the variable a in the HJB-QVI (3.1).

3.2 Compact representation

We now write the GMWB pricing problem in a compact form, which includes the terminal and boundary conditions in a single equation. We define the intervention operator

$$\mathcal{M}(\gamma)v(\mathbf{x}) = \begin{cases} v(w, a-\gamma, \tau) + \gamma(1-\mu) - c & \mathbf{x} \in \Omega^{\infty}_{w_{\min}}, \quad (3.8a) \\ v\left(\ln(\max(e^w - \gamma, e^{w_{-\infty}})), a-\gamma, \tau\right) + \gamma(1-\mu) - c & \mathbf{x} \in \Omega_{\text{in}}. \quad (3.8b) \end{cases}$$

With $\mathbf{x} = (w, a, \tau)$, we let $Dv(\mathbf{x}) = (v_w, v_a, v_\tau)$ and $D^2v(\mathbf{x}) = v_{ww}$, and define

$$F_{\Omega^\infty}(\mathbf{x}, v) \equiv F_{\Omega^\infty}\left(\mathbf{x}, v(\mathbf{x}), Dv(\mathbf{x}), D^2v(\mathbf{x}), \mathcal{J}v(\mathbf{x}), \mathcal{M}v(\mathbf{x})\right), \tag{3.9}$$

[2] There exists a unique viscosity solution in $\left\{\Omega^{\infty}_{w_{\min}} \cup \Omega^{\infty}_{wa_{\min}}\right\} \setminus \{w_{\min}\} \times [a_{\min}, a_{\max}] \times (0, T]$ (see [10, 63]).

where

$$F_{\Omega^\infty}(\mathbf{x}, v) \;=\; \begin{cases} F_{\text{in}}(\mathbf{x}, v) & \equiv \; F_{\text{in}}\big(\mathbf{x}, v(\mathbf{x}), Dv(\mathbf{x}), D^2v(\mathbf{x}), \mathcal{J}v(\mathbf{x}), \mathcal{M}v(\mathbf{x})\big), \quad \mathbf{x} \in \Omega_{\text{in}}, \\ F_{a_{\min}}(\mathbf{x}, v) & \equiv \; F_{a_{\min}}\big(\mathbf{x}, v(\mathbf{x}), Dv(\mathbf{x}), D^2v(\mathbf{x}), \mathcal{J}v(\mathbf{x})\big), \quad \mathbf{x} \in \Omega_{a_{\min}}, \\ F_{w_{\min}}(\mathbf{x}, v) & \equiv \; F_{w_{\min}}(\mathbf{x}, v(\mathbf{x}), Dv(\mathbf{x}), \mathcal{M}v(\mathbf{x})), \quad \mathbf{x} \in \Omega^\infty_{w_{\min}}, \\ F_{wa_{\min}}(\mathbf{x}, v) & \equiv \; F_{wa_{\min}}(\mathbf{x}, v(\mathbf{x}), Dv(\mathbf{x})), \quad \mathbf{x} \in \Omega^\infty_{wa_{\min}}, \\ F_{w_{\max}}(\mathbf{x}, v) & \equiv \; F_{w_{\max}}(\mathbf{x}, v(\mathbf{x})), \quad \mathbf{x} \in \Omega^\infty_{w_{\max}}, \\ F_{\tau_0}(\mathbf{x}, v) & \equiv \; F_{\tau_0}(\mathbf{x}, v(\mathbf{x})), \quad \mathbf{x} \in \Omega^\infty_{\tau_0}, \end{cases}$$

with operators

$$F_{\text{in}}(\mathbf{x}, v) \;=\; \min\left[v_\tau - \mathcal{L}v - \mathcal{J}v - \sup_{\hat{\gamma}\in[0,C_r]} \hat{\gamma}\left(1 - e^{-w}v_w - v_a\right)\mathbf{1}_{\{a>0\}},\; v - \sup_{\gamma\in[0,a]} \mathcal{M}(\gamma)v\right], \tag{3.10}$$

$$F_{a_{\min}}(\mathbf{x}, v) \;=\; v_\tau - \mathcal{L}v - \mathcal{J}v, \tag{3.11}$$

$$F_{w_{\min}}(\mathbf{x}, v) \;=\; \min\left[v_\tau + rv - \sup_{\hat{\gamma}\in[0,C_r]} \hat{\gamma}\,(1 - v_a)\,\mathbf{1}_{\{a>0\}},\; v - \sup_{\gamma\in[0,a]} \mathcal{M}(\gamma)v\right], \tag{3.12}$$

$$F_{wa_{\min}}(\mathbf{x}, v) \;=\; v_\tau + rv, \tag{3.13}$$

$$F_{w_{\max}}(\mathbf{x}, v) \;=\; v - e^{-\beta\tau}(e^w \wedge e^{w_\infty}), \tag{3.14}$$

$$F_{\tau_0}(\mathbf{x}, v) \;=\; v - \max(e^w, (1-\mu)a - c) \wedge e^{w_\infty}. \tag{3.15}$$

Definition 3.1 (Impulse control GMWB pricing problem)**.** *The pricing problem for the GMWB under an impulse control formulation is defined as*

$$F_{\Omega^\infty}\big(\mathbf{x}, v(\mathbf{x}), Dv(\mathbf{x}), D^2v(\mathbf{x}), \mathcal{J}v(\mathbf{x}), \mathcal{M}v(\mathbf{x})\big) \;=\; 0, \tag{3.16}$$

where the operator $F_{\Omega^\infty}(\cdot)$ is defined in (3.9)*.*

We note that F_{Ω^∞} is discontinuous [11, 14] since we include boundary equations in F_{Ω^∞}, which are in general not the limit of the equations from the interior.

Next, we recall the notions of the upper semicontinuous (u.s.c. in short) and the lower semicontinuous (l.s.c. in short) envelops of a function $u : \mathbb{X} \to \mathbb{R}$, where $\mathbb{X}$ is a closed subset of $\mathbb{R}^n$. They are respectively denoted by $u^*(\cdot)$ (for the u.s.c. envelop) and $u_*(\cdot)$ (for the l.s.c. envelop), and are given by

$$u^*(\hat{\mathbf{x}}) = \limsup_{\substack{\mathbf{x}\to\hat{\mathbf{x}} \\ \mathbf{x},\hat{\mathbf{x}}\in\mathbb{X}}} u(\mathbf{x}) \quad (\text{resp.} \quad u_*(\hat{\mathbf{x}}) = \liminf_{\substack{\mathbf{x}\to\hat{\mathbf{x}} \\ \mathbf{x},\hat{\mathbf{x}}\in\mathbb{X}}} u(\mathbf{x})).$$

In general, the solution to impulse control problems are non-smooth, and we seek the viscosity solution of (3.16) [27, 39, 61]. To this end, let $\mathcal{G}(\Omega^\infty)$ be the set of bounded functions defined by [13, 61]

$$\mathcal{G}(\Omega^\infty) = \left\{u : \Omega^\infty \to \mathbb{R}, \quad \sup_{\mathbf{x}\in\Omega^\infty} |u(\mathbf{x})| < \infty\right\}. \tag{3.17}$$

Definition 3.2 (Viscosity solution of equation (3.16))**.** *(i) A locally bounded function $v \in \mathcal{G}(\Omega^\infty)$ is a viscosity subsolution (resp. supersolution) of* (3.16) *in Ω^∞ if for all test function $\phi \in \mathcal{G}(\Omega^\infty) \cap \mathcal{C}^\infty(\Omega^\infty)$ and for all points $\hat{\mathbf{x}} \in \Omega^\infty$ such that $v^* - \phi$ has a* global *maximum on Ω^∞ at $\hat{\mathbf{x}}$ and $v^*(\hat{\mathbf{x}}) = \phi(\hat{\mathbf{x}})$ (resp. $v_* - \phi$ has a* global *minimum on Ω^∞ at $\hat{\mathbf{x}}$ and $v_*(\hat{\mathbf{x}}) = \phi(\hat{\mathbf{x}})$), we have*

$$(F_{\Omega^\infty})_*\big(\hat{\mathbf{x}}, \phi(\hat{\mathbf{x}}), D\phi(\hat{\mathbf{x}}), D^2\phi(\hat{\mathbf{x}}), \mathcal{J}\phi(\hat{\mathbf{x}}), \mathcal{M}\phi(\hat{\mathbf{x}})\big) \;\le\; 0, \tag{3.18}$$

$$\big(\textit{resp.} \quad (F_{\Omega^\infty})^*\big(\hat{\mathbf{x}}, \phi(\hat{\mathbf{x}}), D\phi(\hat{\mathbf{x}}), D^2\phi(\hat{\mathbf{x}}), \mathcal{J}\phi(\hat{\mathbf{x}}), \mathcal{M}\phi(\hat{\mathbf{x}})\big) \;\ge\; 0,\big)$$

where the operator $F_{\Omega^\infty}(\cdot)$ is defined in (3.9)*.*

(ii) A locally bounded function $v \in \mathcal{G}(\Omega^\infty)$ is a viscosity solution of (3.16) *in $\Omega_{in} \cup \Omega_{a_{\min}}$ if v is a viscosity subsolution and a viscosity supersolution in $\Omega_{in} \cup \Omega_{a_{\min}}$.*

Remark 3.1 (Equivalent definitions). *In the existing literature, there are several equivalent definitions of viscosity solution for HJB-QVIs arising from general impulse control problems [27, 61]. Here, equivalence between two different definitions of viscosity solution means that a subsolution (resp. supersolution) in the sense of one definition is also a subsolution (resp. supersolution) in the sense of the other. For example, in Definition 3.2 (i), it is possible to replace* $\phi \in \mathcal{G}(\Omega^\infty) \cap \mathcal{C}^\infty(\Omega^\infty)$ *by* $\phi \in \mathcal{G}(\Omega^\infty) \cap \mathcal{C}^2(\Omega^\infty)$ *[12]. It is also possible to replace* $\phi(\hat{\mathbf{x}})$ *by* $v^*(\hat{\mathbf{x}})$ *(resp.* $v_*(\hat{\mathbf{x}})$*) in the non-local terms* $\mathcal{J}(\cdot)$ *and* $\mathcal{M}(\cdot)$*, as these terms contain no partial derivatives [27]. For the GMWB pricing problem as given in* (3.16)*, equivalence between these definitions can be established (see Appendix B). For the purpose of verifying consistency of a numerical scheme, it is convenient to use Definition 3.2.*

Remark 3.2 (Strong comparison result and convergence region). *Using an equivalent definition of viscosity solution, we can show that the GMWB pricing problem as given in* (3.16) *satisfies a strong comparison principle result in* $\Omega_{in} \cup \Omega_{a_{\min}}$*, where* $\Omega_{a_{\min}} \subset \partial\Omega_{in}$ *(see Appendix B). That is, if* $u_1(\mathbf{x})$ *and* $u_2(\mathbf{x})$ *respectively are subsolution and supersolution in* $\Omega_{in} \cup \Omega_{a_{\min}}$*, of* (3.16)*, then* $u_1(\mathbf{x}) \leq u_2(\mathbf{x})$ *for all* $\mathbf{x} \in \Omega_{in} \cup \Omega_{a_{\min}}$*. Hence, a unique continuous viscosity solution exists in* $\Omega_{in} \cup \Omega_{a_{\min}}$*.*

In general, we cannot hope for a continuous solution to the GMWB problem (3.16) *on all the boundary* $\Gamma = \partial\Omega_{in} \setminus \Omega_{a_{\min}}$ *as it is possible that loss of boundary data can occur over parts of* Γ*, i.e. as* $\tau \to 0$ *and* $w \to \{w_{\min}, w_{\max}\}$ *[40, 58, 65]. However, these problematic parts of* Γ *are trivial in the sense that either the boundary data is used or is irrelevant. In all cases, we consider the computed solution as the limiting value approaching* Γ *from the interior.*

4 Numerical methods

The GMWB pricing problem as given in (3.16) is still posed in an infinite domain, due to the infinite boundary sub-domains in w. For computational purposes, we need to truncate these infinite sub-domains into finite ones. For the purpose of proving convergence, we also need to make sure that the boundary truncation error, i.e. loss of information in the boundary due to this truncation, vanish sufficiently fast as a discretization parameter approaches zero. This is discussed in Subsection 4.1 below.

4.1 Computational domain

A key step of our numerical scheme is a timestepping method based on a convolution integral that involves the Green's function of an associated PIDE in w. In the following, for ease of exposition, we ignore the dependence on a by letting $a \in [a_{\min}, a_{\max}]$ be fixed, and we primarily focus on the dependence on w and τ. Let $\{\tau_m\}$, $m = 0, \ldots, M$, be an equally spaced partition in the τ-dimension, where $\tau_m = m\Delta\tau$ and $\Delta\tau = T/M$. For a fixed $\tau_m > 0$ such that $\tau_{m+1} \leq T$, we consider the PIDE

$$v_\tau - \mathcal{L}v - \mathcal{J}v = 0, \quad w \in (-\infty, \infty), \quad \tau \in (\tau_m, \tau_{m+1}], \tag{4.1}$$

subject to the initial condition at time τ_m given by a function $\hat{v}(w, \cdot, \tau_m)$ where

$$\hat{v}(w, \cdot, \tau_m) = \begin{cases} v_{bc}(w, \cdot, \tau_m) \text{ satisfies (3.7)} & w \in (-\infty, w_{\min}], \\ v(w, \cdot, \tau_m) & w \in (w_{\min}, w_{\max}), \\ v_{bc}(w, \cdot, \tau_m) \text{ satisfies (3.6)} & w \in [w_{\max}, \infty). \end{cases} \tag{4.2}$$

We denote by $g(\cdot)$ the Green's function of the PIDE (4.1) which has the form $g(w, w', \Delta\tau) \equiv g(\, w - w', \Delta\tau\,)$. The solution $v(w, \cdot, \tau_{m+1})$ for $w \in (w_{\min}, w_{\max})$ can be represented as the convolution of $g(\cdot)$ and $\hat{v}(\cdot)$ as follows [30, 36]

$$v(w, \cdot, \tau_{m+1}) = \int_{-\infty}^{\infty} g\left(w - w', \Delta\tau\right)\ \hat{v}(w', \cdot, \tau_m)\ dw', \quad w \in (w_{\min}, w_{\max}). \tag{4.3}$$

The solution $v(w, \cdot, \tau_{m+1})$ for $w \in (-\infty, w_{\min}] \cup [w_{\max}, \infty)$ are given by the boundary conditions (3.6) and (3.7). In the analysis below, we focus on integral (4.3).

For computational purposes, we truncate the infinite interval of integration of (4.3) to $[w^{\dagger}_{\min}, w^{\dagger}_{\max}]$, where $w^{\dagger}_{\min} \ll w_{\min} < 0 < w_{\max} \ll w^{\dagger}_{\max}$ and $|w^{\dagger}_{\min}|$ and $w^{\dagger}_{\max}$ are sufficiently large, resulting in

$$v(w, \cdot, \tau_{m+1}) \;\simeq\; \int_{w^{\dagger}_{\min}}^{w^{\dagger}_{\max}} g(w - w', \Delta\tau)\; \hat{v}(w', \cdot, \tau_m)\; dw', \quad w \in (w_{\min}, w_{\max}). \tag{4.4}$$

We denote by $\mathcal{E}_b$ the error of the above truncation of the integration domain, i.e.

$$\mathcal{E}_b = \int_{\mathbb{R}\setminus[w^{\dagger}_{\min}, w^{\dagger}_{\max}]} g(w - w', \Delta\tau)\; \hat{v}(w', \cdot, \tau_m)\; dw', \quad w \in (w_{\min}, w_{\max}), \tag{4.5}$$

For subsequent use in the paper, let $P^{\dagger} = w^{\dagger}_{\max} - w^{\dagger}_{\min}$. Results in [21][Proposition 4.2] indicate that, for general jump diffusion models, such as those considered in this paper, $\mathcal{E}_b$ is bounded by

$$|\mathcal{E}_b| \;\leq\; K_1 \Delta\tau e^{-K_2 P^{\dagger}}, \quad \forall w \in (w_{\min}, w_{\max}), \quad K_1, K_2 > 0 \text{ independent of } \Delta\tau, P^{\dagger}. \tag{4.6}$$

For fixed $[w^{\dagger}_{\min}, w^{\dagger}_{\max}]$, and hence fixed $P^{\dagger}$, (4.6) shows $\mathcal{E}_b \to 0$, as $\Delta\tau \to 0$. However, as typically required for showing consistency, one would need to ensure $\frac{\mathcal{E}_b}{\Delta\tau} \to 0$ as $\Delta\tau \to 0$. Therefore, from (4.6), we need $P^{\dagger} \to \infty$ as $\Delta\tau \to 0$, which can be achieved by letting $P^{\dagger} = C/\Delta\tau$, for a finite $C > 0$.[3] (For relevant discussions, see, for example, [32][Theorem 4.2]). We note that, for practical purposes, if $P^{\dagger}$ is chosen sufficiently large, it can be kept constant for all $\Delta\tau$ refinement levels (as we let $\Delta\tau \to 0$). The effectiveness of this practical approach is demonstrated through numerical experiments in Section 6.

Remark 4.1 (Padding considerations)**.** *For the PIDE* (4.1)*, the Green's function $g(w, \Delta\tau)$ is not known in closed-form. However, we do have a closed-form representation for the Fourier transform of $g(w, \Delta\tau)$. Therefore, we can approximate* (4.4) *efficiently by discrete convolution via Fast Fourier Transform (FFT). As noted in the introduction, wraparound error (due to periodic extension) is an important issue for Fourier methods, particularly in the case of impulse control problems. For our scheme, the intervals $[w^{\dagger}_{\min}, w_{\min}]$ and $[w_{\max}, w^{\dagger}_{\max}]$ also serve as padding areas for nodes in $\Omega_{in} \cup \Omega_{a_{\min}}$. Without loss of generality, for convenience, we assume that $|w_{\min}|$ and $w_{\max}$ are chosen sufficiently large so that*

$$w^{\dagger}_{\min} = w_{\min} - \frac{w_{\max} - w_{\min}}{2}, \quad \textit{and} \quad w^{\dagger}_{\max} = w_{\max} + \frac{w_{\max} - w_{\min}}{2}. \tag{4.7}$$

In Subsection 4.4, we show that, for practical purposes, this simple choice for padding areas is sufficient for eliminating wraparound error. This is also verified by extensive numerical experiments in Section 6.

We now have a finite computational domain $\Omega = [w^{\dagger}_{\min}, w^{\dagger}_{\max}] \times [a_{\min}, a_{\max}] \times [0, T]$, which consists of

$$\begin{aligned}
&\Omega_{\text{in}} \;\;\;= \text{defined in (3.5)}, && \Omega_{a_{\min}} = \text{defined in (3.5)},\\
&\Omega_{\tau_0} \;\;\;= [w^{\dagger}_{\min}, w^{\dagger}_{\max}] \times [a_{\min}, a_{\max}] \times \{0\}, && \Omega_{w_{\min}} = [w^{\dagger}_{\min}, w_{\min}] \times (a_{\min}, a_{\max}] \times (0, T],\\
&\Omega_{wa_{\min}} = [w^{\dagger}_{\min}, w_{\min}] \times \{a_{\min}\} \times (0, T], && \Omega_{w_{\max}} = [w_{\max}, w^{\dagger}_{\max}] \times [a_{\min}, a_{\max}] \times (0, T].
\end{aligned} \tag{4.8}$$

Due to withdrawals, the non-local impulse operator $\mathcal{M}(\cdot)$ for Ω_{in}, defined in (3.8b), may require evaluating a candidate value at a point having $w = \ln(\max(e^{w} - \gamma, e^{w_{-\infty}}))$, which could be outside $[w^{\dagger}_{\min}, w^{\dagger}_{\max}]$, if $w_{-\infty} < w^{\dagger}_{\min}$. Without loss of generality, we assume $w_{-\infty} \geq w^{\dagger}_{\min}$.

4.2 Discretization

We denote by N (respectively $N^{\dagger}$) the number of points of a uniform partition of $[w_{\min}, w_{\max}]$ (respectively $[w^{\dagger}_{\min}, w^{\dagger}_{\max}]$). For convenience, we typically choose $N^{\dagger} = 2N$ so that only one set of w-coordinates is needed. Recall that $P^{\dagger} = w^{\dagger}_{\max} - w^{\dagger}_{\min}$, and also let $P = w_{\max} - w_{\min}$. We define $\Delta w = \frac{P}{N} = \frac{P^{\dagger}}{N^{\dagger}}$. We use an equally spaced partition in the w-direction, denoted by $\{w_n\}$, where

$$\begin{aligned}
w_n \;&=\; \hat{w}_0 + n\Delta w; \;\; n \;=\; -N^{\dagger}/2, \ldots, N^{\dagger}/2, \;\; \text{where} \\
&\Delta w \;=\; P/N \;=\; P^{\dagger}/N^{\dagger}, \;\; \text{and} \;\; \hat{w}_0 \;=\; (w_{\min} + w_{\max})/2 \;=\; (w^{\dagger}_{\min} + w^{\dagger}_{\max})/2.
\end{aligned} \tag{4.9}$$

[3] For the special case of a GBM, straightforward calculus shows that $|\mathcal{E}_b| \leq C e^{-1/\Delta\tau}/\sqrt{\Delta\tau}$, for a finite $C > 0$, and hence, even with fixed $P^{\dagger}$, we still have $\frac{\mathcal{E}_b}{\Delta\tau} \to 0$, as $\Delta\tau \to 0$.

We use an unequally spaced partition in the a-direction, denoted by $\{a_j\}$, $j = 0, \ldots, J$, with $a_0 = a_{\min}$, and $a_J = a_{\max}$. We use the same previously defined uniform partition $\{\tau_m\}$, $m = 0, \ldots, M$, $\tau_m = m\Delta\tau$ and $\Delta\tau = T/M$.[4] Let $\Delta a_{\max} = \max_j (a_{j+1} - a_j)$, $\Delta a_{\min} = \min_j (a_{j+1} - a_j)$, $j = 0, \ldots, J-1$. In addition, we assume that there is a discretization parameter $h > 0$ such that

$$\Delta w = C_1 h, \quad \Delta a_{\max} = C_2 h, \quad \Delta a_{\min} = C_2' h, \quad \Delta\tau = C_3 h, \quad P^\dagger = C_3'/h, \tag{4.10}$$

where the positive constants C_1, C_2, C_2', C_3 and C_3' are independent of h. We denote by $v_{n,j}^m$ a numerical approximation to the exact solution $v(w_n, a_j, \tau_m)$ at node $(w_n, a_j, \tau_m) \equiv \mathbf{x}_{n,j}^m$. For $m = 1, \ldots, M$, nodes $\mathbf{x}_{n,j}^m$ having (i) $n = -N^\dagger/2, \ldots, -N/2$, are in $\Omega_{w_{\min}} \cup \Omega_{wa_{\min}}$, (ii) $n = -N/2+1, \ldots N/2-1$, are in $\Omega_{\text{in}} \cup \Omega_{a_{\min}}$, and (iii) $n = N/2, \ldots N^\dagger/2$, are in $\Omega_{w_{\max}}$. We conclude this subsection by noting that it is straightforward to ensure the theoretical requirement $P^\dagger \to \infty$ as $h \to 0$. For example, with $C_3' = 1$ in (4.10), we can quadruple $N^\dagger$ as we halve h.

4.3 Numerical scheme

For $(w_n, a_j, \tau_0) \in \Omega_{\tau_0}$, we impose the initial condition (3.15) by

$$v_{n,j}^0 = \max(e^{w_n}, (1-\mu)a_j - c) \wedge e^{w_\infty}, \; n = -N^\dagger/2, \ldots, N^\dagger/2 - 1, \; j = 0, \ldots, J. \tag{4.11}$$

We impose the condition (3.14) for $(w_n, a_j, \tau_{m+1}) \in \Omega_{w_{\max}}$ by

$$v_{n,j}^{m+1} = e^{-\beta\tau_{m+1}}(e^{w_n} \wedge e^{w_\infty}), \; n = N/2, \ldots, N^\dagger/2, \; j = 0, \ldots, J, \; m = 0, \ldots, M-1. \tag{4.12}$$

In the subsequent discussion, we denote by $\gamma_{n,j}^m$ is the control representing the withdrawal amount at node (w_n, a_j, τ_m), $n = -N^\dagger/2, \ldots, N/2-1$, $j = 0, \ldots, J$, $m = 0, \ldots, M-1$. We let $\tau_m^+ = \tau_m + \varepsilon$, $\varepsilon \downarrow 0^+$.

4.3.1 $\Omega_{w_{\min}} \cup \Omega_{wa_{\min}}$

For (w_n, a_j, τ_{m+1}) in $\Omega_{w_{\min}} \cup \Omega_{wa_{\min}}$, let $\tilde{v}_{n,j}^m$ be an approximation to $v(w_n, a_j - \gamma_{n,j}^m, \tau_m)$ computed by linear interpolation. To this end, we denote by $\mathcal{I}\{v^m\}(w, a)$ a two-dimensional linear interpolation operator acting on the time-τ_m discrete solutions $\left\{\left((w_l, a_k), v_{l,k}^m\right)\right\}$, $l = -N^\dagger/2, \ldots, N^\dagger/2$, $k = 0, \ldots, J$, $m = 0, \ldots, M-1$. Then, $\tilde{v}_{n,j}^m$ is computed as follows

$$\tilde{v}_{n,j}^m = \mathcal{I}\{v^m\}\left(w_n, a_j - \gamma_{n,j}^m\right), \quad n = -N^\dagger/2, \ldots, -N/2, \; j = 0, \ldots, J. \tag{4.13}$$

We compute intermediate results $v_{n,j}^{m+}$ by solving

$$v_{n,j}^{m+} = \sup_{\gamma_{n,j}^m \in [0, a_j]} \left(\tilde{v}_{n,j}^m + f\left(\gamma_{n,j}^m\right)\right), \; n = -N^\dagger/2, \ldots, -N/2, \; j = 0, \ldots, J, \tag{4.14}$$

where $\tilde{v}_{n,j}^m$ is given in (4.13) and $f(\cdot)$ is the cash amount received by the investor and is defined by

$$f(\gamma) = \begin{cases} \gamma & \text{if } 0 \le \gamma \le C_r\Delta\tau, \\ \gamma(1-\mu) + \mu C_r \Delta\tau - c & \text{if } C_r\Delta\tau < \gamma. \end{cases} \tag{4.15}$$

To advance to time τ_{m+1}, we solve the first-order ODE $v_\tau + rv = 0$ with the initial condition given by $v_{n,j}^{m+}$ in (4.14) by simply applying a finite difference timestepping method

$$v_{n,j}^{m+1} = v_{n,j}^{m+} - \Delta\tau\left(r v_{n,j}^{m+1}\right), \; n = -N^\dagger/2, \ldots, -N/2, \; j = 0, \ldots, J, \; m = 0, \ldots, M-1. \tag{4.16}$$

We note that (4.16) is strictly monotone. We also emphasize that numerical solutions in $\Omega_{w_{\max}}$ and $\Omega_{w_{\min}} \cup \Omega_{wa_{\min}}$ can be computed without using information from Ω_{in} or $\Omega_{a_{\min}}$.

[4] While it is straightforward to generalize the numerical method to non-uniform partitioning of the τ-dimension, for the purposes of proving convergence, uniform partitioning suffices.

4.3.2 $\Omega_{\text{in}} \cup \Omega_{a_{\min}}$: scheme

For (w_n, a_j, τ_{m+1}) in $\Omega_{\text{in}} \cup \Omega_{a_{\min}}$, let $\tilde{v}_{n,j}^m$ be an approximation to $v(\ln(\max(e^{w_n} - \gamma_{n,j}^m, e^{w_{\min}^\dagger})), a_j - \gamma_{n,j}^m, \tau_m)$ computed by linear interpolation. We compute $\tilde{v}_{n,j}^m$ by linear interpolation as follows

$$\tilde{v}_{n,j}^m = \mathcal{I}\{v^m\}\left(\ln\left(\max\left(e^{w_n} - \gamma_{n,j}^m, e^{w_{\min}^\dagger}\right)\right), a_j - \gamma_{n,j}^m\right), \quad n = -N/2+1, \ldots, N/2-1. \tag{4.17}$$

We note that the $\min\{\cdot\}$ operator of (3.3) contains two terms, with the continuous control $\hat{\gamma}$ in the first term having a local nature ($\hat{\gamma} \in [0, C_r]$), while the impulse control γ in the second term having a non-local nature ($\gamma \in [0, a]$). Motivated by this observation, as in [19], with the convention that $(C_r\Delta\tau, a_j] = \emptyset$ if $a_j \le C_r\Delta\tau$, we partition $[0, a_j]$ into $[0, \min(a_j, C_r\Delta\tau)]$ and $(C_r\Delta\tau, a_j]$. We compute respective intermediate results $(v_{\mathit{loc}})_{n,j}^{m+}$ and $(v_{\mathit{nlc}})_{n,j}^{m+}$ by solving the optimization problems

$$(v_{\mathit{loc}})_{n,j}^{m+} = \sup_{\gamma_{n,j}^m \in [0, \min(a_j, C_r\Delta\tau)]} \left(\tilde{v}_{n,j}^m + f\left(\gamma_{n,j}^m\right)\right), \qquad (v_{\mathit{nlc}})_{n,j}^{m+} = \sup_{\gamma_{n,j}^m \in (C_r\Delta\tau, a_j]} \left(\tilde{v}_{n,j}^m + f\left(\gamma_{n,j}^m\right)\right),$$
$$n = -N/2+1, \ldots, N/2-1, \quad j = 0, \ldots, J, \quad m = 0, \ldots, M-1, \tag{4.18}$$

where $f(\cdot)$ is defined in (4.15) and $\tilde{v}_{n,j}^m$, $n = -N/2+1, \ldots, N/2-1$ is given in (4.17). Intuitively, as $h \to 0$, (v_{loc}) and (v_{nlc}) in (4.18) respectively correspond to the solutions of the first and the second term of the $\min\{\cdot\}$ operator of (3.3) set equal to zero.

Remark 4.2 (Attainability of supremum). *It is straightforward to show that, due to boundedness of nodal values used in $\mathcal{I}\{v^m\}(\cdot)$ (see Lemma 5.1 on stability), the interpolated value $\tilde{v}_{n,j}^m$ in (4.17) is uniformly continuous in $\gamma_{n,j}^m$. As a result, the supremum in the discrete equations for $(v_{\mathit{loc}})_{n,j}^{m+}$ and $(v_{\mathit{nlc}})_{n,j}^{m+}$ in (4.18) can be achieved by a control in $[0, \min(a_j, C_r\Delta\tau)]$ and $(C_r\Delta\tau, a_j]$, respectively, with the latter case being made possible due to $c > 0$ [19].*

To prepare for time advancement to τ_{m+1}, $m = 0, \ldots, M-1$, we combine boundary values $\Omega_{w_{\min}} \cup \Omega_{wa_{\min}}$ and $\Omega_{w_{\max}}$ with results from (4.18) as below (with a slight abuse of notation)

$$\begin{matrix} (v_{\mathit{loc}})_{l,j}^{m+} \\ (\text{resp. } (v_{\mathit{nlc}})_{l,j}^{m+}) \end{matrix} = \begin{cases} v_{l,j}^m & \text{in (4.16)}, \quad l = -N^\dagger/2, \ldots, -N/2, \\ (v_{\mathit{loc}})_{l,j}^{m+} \ (\text{resp. } (v_{\mathit{nlc}})_{l,j}^{m+}) & \text{in (4.18)}, \quad l = -N/2+1, \ldots, N/2-1, \\ v_{l,j}^m & \text{in (4.12)}, \quad l = N/2, \ldots, N^\dagger/2-1. \end{cases} \tag{4.19}$$

For $\tau \in [\tau_m^+, \tau_{m+1}]$, our timestepping method for solving the PIDE (4.1) is the convolution (4.4) with the Green's function being $g(w, \Delta\tau)$ and the initial condition $\hat{v}(w, \cdot, \tau_m^+)$ given by a linear combination of discrete values in (4.19). Specifically, using $(v_{\mathit{loc}})_{l,j}^{m+}$, $l = -N^\dagger/2, \ldots, N^\dagger/2-1$, $\hat{v}(w, \cdot, \tau_m^+)$ is given by

$$\hat{v}\left(w, \cdot, \tau_m^+\right) = \sum_{l=-N^\dagger/2}^{N^\dagger/2-1} \varphi_l(w)\, (v_{\mathit{loc}})_{l,j}^{m+}, \quad w \in [w_{\min}^\dagger, w_{\max}^\dagger]. \tag{4.20}$$

Here, $\{\varphi_l(w)\}$, $l = -N^\dagger/2, \ldots, N^\dagger/2-1$, are piecewise linear basis functions defined by[5]

$$\varphi_l(w) = \begin{cases} (w - w_{l-1})/\Delta w, & w_{l-1} \le w \le w_l, \\ (w_{l+1} - w)/\Delta w, & w_l \le w \le w_{l+1}, \\ 0, & \text{otherwise.} \end{cases} \tag{4.21}$$

The timestepping results $(v_{\mathit{loc}})_{n,j}^{m+1}$, $n = -N/2+1, \ldots, N/2-1$, is given by the discrete convolution

$$(v_{\mathit{loc}})_{n,j}^{m+1} = \int_{w_{\min}^\dagger}^{w_{\max}^\dagger} g(w_n - w, \Delta\tau)\ \hat{v}(w, \cdot, \tau_m^+)\, dw = \Delta w \sum_{l=-N^\dagger/2}^{N^\dagger/2-1} \tilde{g}(w_n - w_l, \Delta\tau)\, (v_{\mathit{loc}})_{l,j}^{m+}, \tag{4.22}$$

$$\text{where} \quad \tilde{g}_{n-l} \equiv \tilde{g}(w_n - w_l, \Delta\tau) = \frac{1}{\Delta w}\int_{w_{\min}^\dagger}^{w_{\max}^\dagger} \varphi_l(w)\ g(w_n - w, \Delta\tau)\, dw. \tag{4.23}$$

[5] For a discussion of different choices of basis functions, see [35].

Using similar steps on $(v_{nlc})_{l,j}^{m+}$, $l = -N^\dagger/2, \ldots, N^\dagger/2 - 1$, in (4.19), gives us the timestepping results $(v_{nlc})_{n,j}^{m+1}$, $n = -N/2+1, \ldots, N/2-1$, $j = 0, \ldots, J$, and $m = 0, \ldots, M-1$.

That is, with $\tilde{g}_{n-l}$ given in (4.23) we compute two discrete convolutions

$$(v_{loc})_{n,j}^{m+1} = \Delta w \sum_{l=-N^\dagger/2}^{N^\dagger/2-1} \tilde{g}_{n-l}\ (v_{loc})_{l,j}^{m+}, \quad (v_{nlc})_{n,j}^{m+1} = \Delta w \sum_{l=-N^\dagger/2}^{N^\dagger/2-1} \tilde{g}_{n-l}\ (v_{nlc})_{l,j}^{m+}. \tag{4.24}$$
$$n = -N/2+1, \ldots, N/2-1,\ j = 0, \ldots, J,\ m = 0, \ldots, M-1.$$

Finally, we compute $v_{n,j}^{m+1}$ by

$$v_{n,j}^{m+1} = \max\left((v_{loc})_{n,j}^{m+1},\ (v_{nlc})_{n,j}^{m+1}\right), \quad \text{where } (v_{loc})_{n,j}^{m+1} \text{ and } (v_{nlc})_{n,j}^{m+1} \text{ from (4.24)},$$
$$n = -N/2+1, \ldots, N/2-1,\ j = 0, \ldots, J,\ m = 0, \ldots, M-1. \tag{4.25}$$

In (4.25), the exact weight $\tilde{g}_{n-l}$, $n = -N/2+1, \ldots, N/2-1$, $l = -N^\dagger/2, \ldots, N^\dagger/2-1$, defined in (4.23), is strictly positive. Therefore, scheme (4.25) is strictly monotone. However, since a closed-form representation for $g(w, \Delta\tau)$ is not known, the exact weight $\tilde{g}_{n-l}$ can only approximated, and hence, this potentially results in negative weights, i.e. loss of monotonicity. In the next subsection, we will show that it is possible to achieve monotonicity, for fixed N and $\Delta\tau$, for any tolerance $\epsilon > 0$.

Remark 4.3 (Optimization method). *In (4.18), we discretize the control $\gamma_{n,j}^m$ with spacing $O(h)$, and solve the optimization problem at each node by exhaustive search, using binary search to query the database of discrete solution values on the unequally spaced (w, a) mesh. As has been proven in [19, Proposition 1], the error in this step is $\mathcal{O}(h^2)$ for any smooth test function. One dimensional optimization methods could be used to reduce the computational cost, but there is then no guarantee of obtaining the global maximum as $h \to 0$.*

4.3.3 $\Omega_{\text{in}} \cup \Omega_{a_{\min}}$: ϵ-monotonicity

To approximate $\tilde{g}_{n-l}$, we follow the same steps as in [35]. For the sake of completeness, we provide some key steps below. We recall the Fourier transform and inverse Fourier transform

$$\mathcal{F}[g(\cdot)] = G(\eta, \Delta\tau) = \int_{-\infty}^{\infty} e^{-2\pi i\eta w} g(w, \Delta\tau) dw, \quad \mathcal{F}^{-1}[G(\cdot)] = g(w, \Delta\tau) = \int_{-\infty}^{\infty} e^{2\pi i\eta w} G(\eta, \Delta\tau) d\eta. \tag{4.26}$$

It is straightforward to show that a closed-form expression for $G(\eta, \Delta\tau)$, the Fourier transform of the Green's function of equation (4.1), is

$$G(\eta, \Delta\tau) = \exp(\Psi(\eta)\Delta\tau), \quad \text{with}$$
$$\Psi(\eta) = \left(-\frac{1}{2}\sigma^2(2\pi\eta)^2 + \left(r - \lambda\kappa - \frac{1}{2}\sigma^2 - \beta\right)(2\pi i\eta) - (r+\lambda) + \lambda\overline{B}(\eta)\right). \tag{4.27}$$

Here, $\overline{B}(\eta)$ is the complex conjugate of the integral $B(\eta) = \int_{-\infty}^{\infty} b(y)\ e^{-2\pi i\eta y}\ dy$, noting $b(y)$ is the density function of $\ln(\psi)$, where ψ is the random variable representing the jump multiplier.

For a fixed $n \in \{-N/2+1, \ldots, N/2-1\}$, to approximate $\tilde{g}_{n-l}$, $l = -N^\dagger/2, \ldots, N^\dagger/2-1$, in (4.23), we replace $g(w, \Delta\tau)$ by its localized, periodic approximation $\hat{g}(w, \Delta\tau)$ given by

$$\hat{g}(w, \Delta\tau) = \frac{1}{P^\dagger} \sum_{k=-\infty}^{\infty} e^{2\pi i\eta_k w} G(\eta_k, \Delta\tau) \quad \text{with} \quad \eta_k = \frac{k}{P^\dagger}, \quad P^\dagger = w_{\max}^\dagger - w_{\min}^\dagger. \tag{4.28}$$

Remark 4.4. *We note that the coefficients $G(\eta_k, \Delta\tau)$ in (4.28) are the exact coefficients corresponding to the Green's function of the PIDE (4.1) with periodic boundary conditions at $w_{\min}^\dagger$ and $w_{\max}^\dagger$. Hence, $\hat{g}(w, \Delta\tau)$ is a valid Green's function, and in particular $\hat{g}(\cdot) \geq 0$.*

We note that, for a fixed $\Delta\tau$, $\hat{g}(w, \Delta\tau) \neq g(w, \Delta\tau)$, $w \in [w_{\min}^\dagger, w_{\max}^\dagger]$. However, as $\Delta\tau \to 0$, or equivalently, as $h \to 0$, we have

$$\hat{g}(w, \Delta\tau) \overset{(i)}{=} \int_{-\infty}^{\infty} e^{2\pi i\eta w} G(\eta, \Delta\tau) d\eta + \mathcal{O}\left(1/(P^\dagger)^2\right) \underset{(4.26)}{\overset{by}{=}} g(w, \Delta\tau) + \mathcal{O}\left(h^2\right). \tag{4.29}$$

Here, (i) is due to $P^\dagger \to \infty$ as $h \to 0$, ensuring in an $\mathcal{O}\left(1/(P^\dagger)^2\right) \sim \mathcal{O}\left(h^2\right)$ error for the traperzoidal rule approximation of the integral.

After replacing $g(w, \Delta\tau)$ by $\hat{g}(w, \Delta\tau)$ in (4.23), we integrate the resulting finite integral and obtain

$$\tilde{g}_{n-l} \;\equiv\; \tilde{g}_{n-l}(\infty) \;=\; \frac{1}{P^\dagger}\left(\sum_{k=-\infty}^{\infty} e^{2\pi i \eta_k (n-l)\Delta w}\left(\frac{\sin^2 \pi\eta_k\Delta w}{(\pi\eta_k\Delta w)^2}\right)\, G(\eta_k, \Delta\tau)\right). \tag{4.30}$$

For $\alpha \in \{2, 4, 8, \ldots\}$, (4.30) is truncated to $\alpha N^\dagger$ terms, resulting in an approximate weight

$$\tilde{g}_{n-l}(\alpha) = \frac{1}{P^\dagger}\left(\sum_{k=-\alpha N^\dagger/2}^{\alpha N^\dagger/2-1} e^{2\pi i \eta_k (n-l)\Delta w}\left(\frac{\sin^2 \pi\eta_k\Delta w}{(\pi\eta_k\Delta w)^2}\right)\, G(\eta_k, \Delta\tau)\right). \tag{4.31}$$

As $\alpha \to \infty$, there is no loss of information in the discrete convolution (4.31). However, for any finite α, there is an error due to the use of a truncated Fourier series, which is shown to be [35]

$$\tilde{g}_{n-l}(\alpha) - \tilde{g}_{n-l}(\infty) = \mathcal{O}(e^{-1/h}). \tag{4.32}$$

Although the error in (4.32) indicates a rapid convergence of truncated Fourier series as $\alpha \to \infty$, strict monotonicity is not guaranteed for a finite α. To control this potential loss of monotonicity for a finite α, as in [35], the selected α must satisfy

$$\Delta w \sum_{l=-N^\dagger/2}^{N^\dagger/2-1} \left|\min\left(\tilde{g}_{n-l}(\alpha), 0\right)\right| \;<\; \epsilon\frac{\Delta\tau}{T}, \quad \forall n \in \{-N/2+1, \ldots, N/2-1\}, \tag{4.33}$$

where $0 < \epsilon \ll 1$ is an user-defined monotonicity tolerance. As discussed in detail in Section 5, to show convergence of the numerical scheme, we need $\epsilon \to 0$ as $h \to 0$. In practice, however, if ϵ is chosen sufficiently small, it can be kept constant for all refinement levels (as we let $h \to 0$). The effectiveness of this practical approach is demonstrated through numerical experiments in Section 6.

4.3.4 Efficient implementation via FFT and algorithms

For a fixed $\alpha \in \{2, 4, 8, \ldots\}$, the sequence $\{\tilde{g}_{-N^\dagger/2}(\alpha), \ldots, \tilde{g}_{N^\dagger/2-1}(\alpha)\}$ is $N^\dagger$-periodic. With this in mind, we let $q = n - l$ in the discrete convolution (4.31), and, for a fixed α, the set of approximate weights in the physical domain to be determined is $\tilde{g}_q(\alpha)$, $q = -N^\dagger/2, \ldots, N^\dagger/2-1$. Using this notation, in (4.31), with $q = n - l$, we rewrite $e^{2\pi i \eta_k (n-l)\Delta w} = e^{2\pi i k \alpha q/(\alpha N^\dagger)}$, and obtain

$$\begin{aligned} \tilde{g}_q(\alpha) &= \frac{1}{P^\dagger}\sum_{k=-\alpha N^\dagger/2}^{\alpha N^\dagger/2-1} e^{2\pi i k(\alpha q)/(\alpha N^\dagger)}\; y_k, \qquad q = -N^\dagger/2, \ldots, N^\dagger/2-1, \\ \text{where } y_k &= \left(\frac{\sin^2 \pi\eta_k\Delta w}{(\pi\eta_k\Delta w)^2}\right) G(\eta_k, \Delta\tau), \quad k = -\frac{\alpha N^\dagger}{2}, \ldots, \frac{\alpha N^\dagger}{2} - 1. \end{aligned} \tag{4.34}$$

It is observed from (4.34) that, given $\{y_k\}$, $\{\tilde{g}_q(\alpha)\}$ can be computed efficiently via a single FFT of size $\alpha N^\dagger$. A suitable value for α, i.e. satisfying the ϵ-monotonicity condition (4.33), can be determined through an iterative procedure based on formula (4.34). Let this value be α_ϵ. We also observe that, once α_ϵ is found, the discrete convolutions (4.24) can also be computed efficiently using an FFT. This suggests that we only need to compute the weights in the Fourier domain, i.e. the DFT of $\{\tilde{g}_q(\alpha_\epsilon)\}$, only once, and reuse them for all timesteps. We define $\{\tilde{G}_q(\alpha_\epsilon)\}$ to be the DFT of $\{\tilde{g}_q(\alpha_\epsilon)\}$ given by

$$\tilde{G}(\eta_k, \Delta\tau, \alpha_\epsilon) = \frac{P^\dagger}{N^\dagger}\sum_{q=-N^\dagger/2}^{N^\dagger/2-1} e^{-2\pi i q k/N^\dagger}\; \tilde{g}_q(\alpha_\epsilon), \quad k = -N^\dagger/2, \ldots, N^\dagger/2-1. \tag{4.35}$$

An iterative procedure for computing $\{\tilde{G}_q(\alpha_\epsilon)\}$ is given in Algorithm 4.1, where we also use the stopping criterion $\Delta w \sum_{q=-N^\dagger/2}^{N^\dagger/2-1} \left|\tilde{g}_q(\alpha) - \tilde{g}_q(\alpha/2)\right| < \epsilon_1$, $\epsilon_1 > 0$. As noted in [35], Algorithm 4.1 stops after a finite number of iterations. For practical purposes, α_ϵ is typically 2 or 4.

Algorithm 4.1 Computation of weights $\tilde{G}_q(\alpha_\epsilon)$, $q = -N^\dagger/2, \ldots, N^\dagger/2 - 1$, in Fourier domain.

1: set $\alpha = 1$ and compute $\tilde{g}_q(\alpha)$, $q = -N^\dagger/2, \ldots, N^\dagger/2 - 1$ using (4.34);
2: **for** $\alpha = 2, 4, \ldots$ until convergence **do**
3: compute $\tilde{g}_q(\alpha)$ $q = -N^\dagger/2, \ldots, N^\dagger/2 - 1$, using (4.34);
4: compute $\text{test}_1 = \Delta w \sum_{q=-N^\dagger/2}^{N^\dagger/2-1} \min(\tilde{g}_q(\alpha), 0)$ for monotonicity test;
5: compute $\text{test}_2 = \Delta w \sum_{q=-N^\dagger/2}^{N^\dagger/2-1} \left|\tilde{g}_q(\alpha) - \tilde{g}_q(\alpha/2)\right|$ for accuracy test;
6: **if** $|\text{test}_1| < \epsilon(\Delta\tau/T)$ and $\text{test}_2 < \epsilon_1$ **then**
7: $\alpha_\epsilon = \alpha$;
break from for loop;
8: **end if**
9: **end for**
10: use (4.35) to compute and output weights $\tilde{G}_q(\alpha_\epsilon)$, $q = -N^\dagger/2, \ldots, N^\dagger/2 - 1$, in Fourier domain.

Remark 4.5. *For simplicity, unless otherwise stated, we adopt the notional convention* $\tilde{g}_{n-l} = \tilde{g}_{n-l}(\alpha_\epsilon)$ *and* $\tilde{G}(\eta_k, \Delta\tau) \equiv \tilde{G}(\eta_k, \Delta\tau, \alpha_\epsilon)$*, where* α_ϵ *is selected by Algorithm 4.1, hence satisfies the* ϵ*-monotonicity condition* (4.33)*:* $\Delta w \sum_{l=-N^\dagger/2}^{N^\dagger/2-1} \left|\min(\tilde{g}_{n-l}(\alpha), 0)\right| < \epsilon \frac{\Delta\tau}{T}$*,* $\epsilon > 0$*, for all* $n \in \{-N/2+1, \ldots, N/2-1\}$*.*

The discrete convolutions (4.24) can then be implemented efficiently via an FFT as follows

$$(v_{loc})_{n,j}^{m+1} \simeq \sum_{q=-N^\dagger/2}^{N^\dagger/2-1} e^{2\pi i q n/N^\dagger} \; V_{loc}(\eta_q, a_j, \tau_m^+) \; \tilde{G}(\eta_q, \Delta\tau), \tag{4.36}$$

$$\text{with } V_{loc}\left(\eta_q, a_j, \tau_m^+\right) = \frac{1}{N^\dagger} \sum_{l=-N^\dagger/2}^{N^\dagger/2-1} e^{-2\pi i q l/N^\dagger} \; (v_{loc})_{l,j}^{m+}, \; q = -N^\dagger/2, \ldots, N^\dagger/2 - 1,$$

where $\tilde{G}(\eta_q, \Delta\tau)$ is given by (4.35). Similarly, we can compute $(v_{nlc})_{n,j}^{m+1}$, $n = -N/2+1, \ldots, N/2-1$, $j = 0, \ldots, J$, and $m = 0, \ldots, M-1$, using an FFT as above. Putting everything together, an ϵ-monotone algorithm for Ω is presented in Algorithm 4.2, where, for simplicity, we use the notation $\mathbb{N}^\dagger = \{-N^\dagger/2, \ldots, N^\dagger/2 - 1\}$.

Algorithm 4.2 An ϵ-monotone Fourier algorithm for GMWB problem defined in Definition (3.1). $x \circ y$ is the Hadamard product of vectors x and y; $\mathbb{N}^\dagger = \{-N^\dagger/2, \ldots, N^\dagger/2 - 1\}$.

1: compute vector $\tilde{G} = \left[\tilde{G}(\eta_q, \Delta\tau)\right]_{q \in \mathbb{N}^\dagger}$, using Algorithm 4.1;
2: initialize $v_{n,j}^0 = \max(e^{w_n}, (1-\mu)a_j - c)$, $n = -\frac{N^\dagger}{2}, \ldots, \frac{N^\dagger}{2}$, $j = 0, \ldots, J$;
3: **for** $m = 0, \ldots, M-1$ **do**
4: solve (4.18) to obtain $(v_{loc})_{n,j}^{m+}$ and $(v_{nlc})_{n,j}^{m+}$, $n = -\frac{N}{2}+1, \ldots, \frac{N}{2}-1$, $j = 0, \ldots, J$; //$\Omega_{\text{in}} \cup \Omega_{a_{\min}}$
5: combine results in Line-4 with $v_{n,j}^m$ in $\Omega_{w_{\min}}$, $\Omega_{wa_{\min}}$ and $\Omega_{w_{\max}}$, to obtain vectors $(v_{loc})_j^{m+} = \left[(v_{loc})_{n,j}^{m+}\right]_{n \in \mathbb{N}^\dagger}$ and $(v_{nlc})_j^{m+} = \left[(v_{nlc})_{n,j}^{m+}\right]_{n \in \mathbb{N}^\dagger}$, $j = 0, \ldots, J$;
6: compute vectors $\left[(v_{loc})_{n,j}^{m+1}\right]_{n \in \mathbb{N}^\dagger} = \text{IFFT}\left\{\text{FFT}\left\{(v_{loc})_j^{m+}\right\} \circ \tilde{G}\right\}$, $j = 0, \ldots, J$;
7: compute vectors $\left[(v_{nlc})_{n,j}^{m+1}\right]_{n \in \mathbb{N}^\dagger} = \text{IFFT}\left\{\text{FFT}\left\{(v_{nlc})_j^{m+}\right\} \circ \tilde{G}\right\}$, $j = 0, \ldots, J$;
8: discard FFT values in $\Omega_{w_{\min}}$, $\Omega_{wa_{\min}}$ and $\Omega_{w_{\max}}$, namely $(v_{loc})_{n,j}^{m+1}$ and $(v_{nlc})_{n,j}^{m+1}$, $n = -\frac{N^\dagger}{2}, \ldots, -\frac{N}{2}$, and $n = \frac{N}{2}, \ldots, \frac{N^\dagger}{2} - 1$, $j = 0, \ldots, J$;
9: set $v_{n,j}^{m+1} = \max\left((v_{loc})_{n,j}^{m+1}, (v_{nlc})_{n,j}^{m+1}\right)$, $n = -\frac{N}{2}+1, \ldots, \frac{N}{2}-1$, $j = 0, \ldots, J$; //$\Omega_{\text{in}} \cup \Omega_{a_{\min}}$
10: compute $v_{n,j}^{m+1}$, $n = \frac{N}{2}, \ldots, \frac{N^\dagger}{2}$, $j = 0, \ldots, J$, using (4.12); //$\Omega_{w_{\max}}$
11: compute $v_{n,j}^{m+1}$, $n = -\frac{N^\dagger}{2}, \ldots, -\frac{N}{2}$, $j = 0, \ldots, J$, using (4.16); //$\Omega_{w_{\min}} \cup \Omega_{wa_{\min}}$
12: **end for**

Remark 4.6 (Algorithm complexity)**.** *The complexity of Algorithm 4.2, at each timestep, consists of two major parts, intervention action and time advancement. For intervention action, a binary search is carried out for each mesh node, with each search costing $\mathcal{O}(|\log(1/h)|)$. For each timestep, we need to solve $\mathcal{O}(1/h^2)$ optimization problems (that is, for each mesh node (w_n, a_j) with $n = -\frac{N^\dagger}{2}, \ldots, \frac{N}{2} - 1$, $j = 0, \ldots, J$), each optimization performs $\mathcal{O}(1/h)$ linear interpolations (i.e. for $\mathcal{O}(1/h)$ elements in the admissible control set). The intervention action results in $\mathcal{O}(|\log(1/h)|/h^3)$ computational cost at each timestep. Regarding time advancement, we basically solve $\mathcal{O}(1/h)$ PIDEs (i.e. for each a_j when $j = 0, \ldots, J$) using the ϵ-monotone Fourier method. Apart from a preprocessing step in Algorithm 4.1, the complexity of the time advancement mainly depends on the FFT to evaluate the discrete convolution, with each FFT costing $\mathcal{O}(|\log(1/h)|/h)$. In total, the computational cost of the time advancement is $\mathcal{O}(|\log(1/h)|/h^2)$ at each timestep. Thus the major cost of Algorithm 4.2 is determined by the intervention action, that is by the local optimization problems.*

4.4 Wraparound error

A well-known issue requiring special attention is that FFT algorithms effectively assumes that the input functions are periodic. This tends to cause wraparound pollution near the boundaries, unless special care is taken when implementing the algorithms [29]. In our case, wraparound error may occur at nodes near $w_{\min}$ and $w_{\max}$, i.e. near the boundaries between $\Omega_{\text{in}} \cup \Omega_{a_{\min}}$ and $\Omega_{w_{\min}} \cup \Omega_{wa_{\min}}$ or $\Omega_{w_{\max}}$, with the contamination being particularly problematic near $w_{\min}$. This is because the non-local impulse operator always moves the solution to smaller w values, due to withdrawals.

As introduced in Remark 4.1, the boundary sub-domains $\Omega_{w_{\min}} \cup \Omega_{wa_{\min}}$ and $\Omega_{w_{\max}}$ are also set up to act as padding areas to minimize the wraparound error in the computation of discrete convolutions (4.24) via an FFT in (4.36). Specifically, as stated in Algorithm 4.2, for each τ_m, solutions in the boundary sub-domains $\Omega_{w_{\min}} \cup \Omega_{wa_{\min}}$ and $\Omega_{w_{\max}}$ are combined with $(v_{loc})_{n,j}^{m+}$ and $(v_{nlc})_{n,j}^{m+}$ in $\Omega_{\text{in}} \cup \Omega_{a_{\min}}$ (Lines 4-5) to form the data for an FFT (Lines 6-7). After an FFT is applied, all results of auxiliary padding nodes in $\Omega_{w_{\min}} \cup \Omega_{wa_{\min}}$ and $\Omega_{w_{\max}}$ are discarded to minimize the wraparound error at nodes in $\Omega_{\text{in}} \cup \Omega_{a_{\min}}$ (Line 8). Note that our treatment is different from the zero padding technique used in [1, 45], which might produce errors near $w_{\min}$. In the below, we show that, with our choice of $N^\dagger = 2N$, N is chosen large enough, our handling of wraparound described above is sufficiently effective.

For full generality, we consider the generic recursion in the form of the discrete convolution (4.24)

$$u_n^{m+1} = \Delta w \sum_{l=-N^\dagger/2}^{N^\dagger/2-1} \tilde{g}_{n-l}\; u_l^m, \quad n = -N/2+1, \ldots, N/2-1. \tag{4.37}$$

As noted above, wraparound in (4.37) may occur if $(n-l) < -N^\dagger/2$ or $(n-l) > N^\dagger/2 - 1$. (Also see Appendix A.) This leads us to the following formal definition of wraparound error at each time τ_m.

Definition 4.1 (wraparound error)**.** *Assume $\{\tilde{g}_q\}$, $q = -N^\dagger/2, \ldots, N^\dagger/2-1$, is periodic with period $N^\dagger$ and u_l^m, for $l < -N/2+1$ or $l > N/2-1$, are determined by boundary data with $N^\dagger = 2N$. Then, the wraparound error for equation (4.37), at timestep m, denoted by e_{wrap}^m, is*

$$e_{wrap}^m \;=\; \max_{-N/2+1 \le n \le N/2-1} \quad \sum_{l=-N^\dagger/2}^{N^\dagger/2-1} \left|\tilde{g}_{n-l}\; u_l^m\right| \left(\mathbf{1}_{\{(n-l)<-N^\dagger/2\}} + \mathbf{1}_{\{(n-l)>N^\dagger/2-1\}}\right).$$

We now state a theorem on the effectiveness of our padding technique. See Appendix A for a proof.

Theorem 4.1. *Let $\{\tilde{g}_q\}$, $q = -N^\dagger/2, \ldots, N^\dagger/2-1$, be periodic with period $N^\dagger$, and u_l^m, for $l < -N/2+1$ or $l > N/2-1$, be determined by boundary data with $N^\dagger = 2N$. Assume further that $\{u_l^m\}$ is bounded in ℓ_∞-norm, so that for $0 \le m \le M$, there exists a constant $C > 0$ such that*

$$|u_l^m| \le C, \quad l \;=\; -N^\dagger/2, \ldots, N^\dagger/2-1. \tag{4.38}$$

If N is selected sufficiently large so that

$$\Delta w \sum_{l=-N^\dagger/2}^{-N/2} |\tilde{g}_l| \;\le\; \frac{\epsilon_e}{2}\Delta\tau \quad \textit{and} \quad \Delta w \sum_{l=N/2}^{N^\dagger/2-1} |\tilde{g}_l| \;\le\; \frac{\epsilon_e}{2}\Delta\tau\ , \quad \epsilon_e > 0, \tag{4.39}$$

then the wraparound error after M steps is bounded by $TC\epsilon_e$.

We now have a corollary about the wraparound error of our scheme.

Corollary 4.1. *The wraparound error, defined in Definition 4.1, of scheme* (4.11), (4.12), (4.16), *and* (4.25), *is bounded by $TC\epsilon_e$, where $\epsilon_e > 0$ can be made arbitrarily small by choosing N sufficiently large.*

5 Convergence to the viscosity solution

It is established by Barles-Souganidis in [14] that, provided a comparison result for PDEs applies, a numerical scheme converges to the unique viscosity solution of the equation if the scheme is ℓ_∞-stable, strictly monotone, and consistent. In our case, as noted in Remark 3.2, a provable strong comparison principle result exists for $\Omega_{\text{in}} \cup \Omega_{a_{\min}}$. However, our scheme is only monotone within a tolerance $\epsilon > 0$ (see (4.33)), and hence, the framework in [14] is not directly applicable. Nonetheless, [14] does note that the monotonicity requirement can be relaxed. This idea was explored in [17].

In this section, we appeal to a Barles-Souganidis-type analysis to rigorously study the convergence of our scheme in $\Omega_{\text{in}} \cup \Omega_{a_{\min}}$ as $h \to 0$ by verifying three properties: ℓ_∞-stability, ϵ-monotonicity (as opposed to strict monotonicity), and consistency. We will show that convergence of our scheme is ensured if the monotonicity tolerance $\epsilon \to 0$ as $h \to 0$. Although our proofs share some similarities with those in [19] for a strictly monotone scheme, we stress that these are distant similarities. Specifically, due to key differences in the monotonicity property and the use of Fourier methods which requires careful handling of boundary regions, our proof techniques are significantly more involved. We will emphasize these key differences where suitable.

For subsequent use, we state two results below: for any $n \in \{-N/2+1, \ldots, N/2-1\}$, we have

$$\Delta w \sum_{l=-N^\dagger/2}^{N^\dagger/2-1} \tilde{g}_{n-l} = e^{-r\Delta\tau}, \quad \Delta w \sum_{l=-N^\dagger/2}^{N^\dagger/2-1} \left(\max\left(\tilde{g}_{n-l}, 0\right) + \left|\min\left(\tilde{g}_{n-l}, 0\right)\right|\right) \le 1 + 2\epsilon\frac{\Delta\tau}{T} \le e^{2\epsilon\frac{\Delta\tau}{T}}. \tag{5.1}$$

Here, the first result is proved in [35], while the second follows from the first, noting $e^{-r\Delta\tau} \le 1$, $\tilde{g}_{n-l} = \max(\tilde{g}_{n-l}, 0) + \min(\tilde{g}_{n-l}, 0)$, together with the monotonicity condition (4.33).

Our scheme consists of the following equations: (4.11) for Ω_{τ_0}, (4.12) for $\Omega_{w_{\max}}$, (4.16) for $\Omega_{w_{\min}} \cup \Omega_{wa_{\min}}$, and finally (4.25) for $\Omega_{\text{in}} \cup \Omega_{a_{\min}}$. We start by verifying ℓ_∞-stability of our scheme.

5.1 Stability

Lemma 5.1 (ℓ_∞-stability)**.** *Suppose the discretization parameter h satisfies* (4.10). *If linear interpolation is used to compute $\tilde{v}_{n,j}^m$ in* (4.13) *and* (4.17), *then scheme* (4.11), (4.12), (4.16), *and* (4.25) *satisfies $\sup_{h>0} \|v^m\|_\infty < \infty$ for all $m = 0, \ldots, M$, as the discretization parameter $h \to 0$. Here, $\|v^m\|_\infty = \max_{n,j} |v_{n,j}^m|$, $n = -N^\dagger/2, \ldots, N^\dagger/2-1$, and $j = 0, \ldots, J$.*

Proof. We note that, for any fixed $h > 0$, we have $\|v^0\|_\infty < \infty$, and therefore, $\sup_{h>0} \|v^0\|_\infty < \infty$. Motivated by this observation, to demonstrate ℓ_∞-stability of our scheme, we will show that, for a fixed $h > 0$, at any (w_n, a_j, τ_m), we have

$$|v_{n,j}^m| < K(\|v^0\|_\infty + a_j), \qquad K > 0 \text{ bounded above independently of } h. \tag{5.2}$$

Since $a_j \le z_0 < \infty$, where z_0 is the up-front premium to the insurer, (5.2) essentially means that $\|v^m\| \le \infty$ for a fixed $h > 0$. Therefore, we obtain $\sup_{h>0} \|v^m\|_\infty < \infty$ for all $m = 0, \ldots, M$, as wanted. We note that the constant $K > 0$ is typically of the form $e^{2m\epsilon\frac{\Delta\tau}{T}}$, $m = 0, \ldots, M$, where ϵ is the monotonicity tolerance used in (4.33) with $0 < \epsilon \ll 1$. Since $m\Delta\tau \le T$, K is bounded above by e^2.

For the rest of the proof, we will show the key inequality (5.2) when $h > 0$ is fixed. For clarity, we will address stability for the boundary and interior sub-domains (together with their respective initial conditions) separately, starting with the boundary sub-domains. It is straightforward to show that (4.11) and (4.12) are ℓ_∞-stable, since

$$\max_{n,j} |v_{n,j}^m| \leq \|v^0\|_\infty, \quad n = N/2, \ldots N^\dagger/2,\ j = 0, \ldots, J,\ m = 0, \ldots, M. \tag{5.3}$$

Similarly, we can also show ℓ_∞-stability of (4.11) and (4.16) by proving $\max_{n,j} |v_{n,j}^m| \leq \|v^0\|_\infty + a_j$ via

$$0 \leq v_{n,j}^m \leq \|v^0\|_\infty + a_j, \quad n = -N^\dagger/2, \ldots - N/2,\ j = 0, \ldots, J,\ m = 0, \ldots, M. \tag{5.4}$$

This can be done by induction on m in a straightforward manner, noting that (4.11) and (4.16) are strictly monotone. We omit this for brevity.

We now prove stability for (4.11) and (4.25). For $n = -N/2+1, \ldots, N/2-1$ and $j = 0, \ldots, J$, and $m = 0, \ldots, M$, we define the measures

$$\left\|v_j^{m+}\right\|_\infty = \max_n \left|v_{n,j}^{m+}\right| \quad \text{and} \quad \|v_j^m\|_\infty = \max_n \left|v_{n,j}^m\right|, \text{ where}$$

$$\left[v_j^{m+}\right]_{\max} = \max_n \left\{v_{n,j}^{m+}\right\},\ \left[v_j^m\right]_{\max} = \max_n \left\{v_{n,j}^m\right\},\ \left[v_j^{m+}\right]_{\min} = \min_n \left\{v_{n,j}^{m+}\right\},\ \left[v_j^m\right]_{\min} = \min_n \left\{v_{n,j}^m\right\}.$$

Similarly, we also have $\left\|(v_{loc})_j^m\right\|_\infty$ and $\left\|(v_{nlc})_j^m\right\|_\infty$, and other respective measures.

Recall the monotonicity tolerance ϵ, where $0 < \epsilon \ll 1$, used in (4.33). To prove stability for (4.11) and (4.25), we show that, for $m \in \{0, \ldots M\}$, we have

$$\|v_j^m\|_\infty \leq e^{2m\epsilon \frac{\Delta\tau}{T}} \left(\|v^0\|_\infty + a_j\right), \quad j = 0, \ldots, J, \tag{5.5}$$

which is bounded above by $e^2(\|v^0\|_\infty + z_0)$ independently of h, since $m\Delta\tau \leq T$. We typically use $\epsilon \leq 1/2$ in the proof below. To show (5.5), using induction on m, $m = 0, \ldots, M$, we will show that, for all $j \in \{0, \ldots, J\}$,

$$\left[v_j^m\right]_{\max} \leq e^{2m\epsilon \frac{\Delta\tau}{T}} \left(\|v^0\|_\infty + a_j\right), \tag{5.6}$$

$$-2m\epsilon \frac{\Delta\tau}{T} e^{2m\epsilon \frac{\Delta\tau}{T}} \left(\|v^0\|_\infty + a_j\right) \leq \left[v_j^m\right]_{\min}. \tag{5.7}$$

We note that numerical solutions at nodes in $\Omega_{w_{\min}} \cup \Omega_{wa_{\min}}$ satisfy the bounds (5.6)-(5.7) at the same $j \in \{j = 0, \ldots, J\}$ and $m \in \{0, \ldots, M\}$,

$$\max_{-N^\dagger/2 \leq n \leq -N/2} \left\{v_{n,j}^m\right\} \text{ satisfies (5.6)}, \quad \text{and} \quad \min_{-N^\dagger/2 \leq n \leq -N/2} \left\{v_{n,j}^m\right\} \text{ satisfies (5.7)}. \tag{5.8}$$

<u>Base case:</u> when $m = 0$, (5.6)-(5.7) hold for all $j \in \{0, \ldots, J\}$, which follows from the initial condition (4.11) for $n = -N/2+1, \ldots, N/2-1$.

<u>Hypothesis:</u> we assume that (5.6)-(5.7) hold for $m = \hat{m}$, where $\hat{m} \leq M-1$, and $n = -N/2+1, \ldots, N/2-1$, $j = 0, \ldots, J$.

<u>Induction:</u> we show that (5.6)-(5.7) also hold for $m = \hat{m}+1$ and $j = 0, \ldots, J$. This is done in two steps. In Step 1, we show, for $j = 0, \ldots, J$,

$$\left[v_j^{\hat{m}+}\right]_{\max} \leq e^{2\hat{m}\epsilon \frac{\Delta\tau}{T}} \left(\|v^0\|_\infty + a_j\right) \tag{5.9}$$

$$-2\hat{m}\epsilon \frac{\Delta\tau}{T} e^{2\hat{m}\epsilon \frac{\Delta\tau}{T}} \left(\|v^0\|_\infty + a_j\right) \leq \left[v_j^{\hat{m}+}\right]_{\min}. \tag{5.10}$$

In Step 2, we bound the timestepping result (4.25) at $m = \hat{m}+1$ using (5.9)-(5.10).

<u>Step 1 - Bound for $v_{n,j}^{\hat{m}+}$:</u> Since $v_{n,j}^{\hat{m}+} = \max\left((v_{loc})_{n,j}^{\hat{m}+}, (v_{nlc})_{n,j}^{\hat{m}+}\right)$, using (4.18), we have

$$v_{n,j}^{\hat{m}+} = \sup_{\gamma_{n,j}^{\hat{m}} \in [0, a_j]} \left[\mathcal{I}\left\{v^{\hat{m}}\right\}\left(\max\left(e^{w_n} - \gamma_{n,j}^{\hat{m}}, e^{w_{\min}^\dagger}\right), a_j - \gamma_{n,j}^{\hat{m}}\right) + f(\gamma_{n,j}^{\hat{m}})\right]. \tag{5.11}$$

As noted in Remark 4.2, for the case $c > 0$ as considered here, the supremum of (5.11) is achieved by an optimal control $\gamma^* \in [0, a_j]$. That is, (5.11) becomes

$$v_{n,j}^{\hat{m}+} = \mathcal{I}\left\{v^{\hat{m}}\right\}\left(\max\left(e^{w_n} - \gamma^*, e^{w_{\min}^\dagger}\right), a_j - \gamma^*\right) + f(\gamma^*), \quad \gamma^* \in [0, a_j]. \tag{5.12}$$

We assume that $\max\left(e^{w_n}-\gamma^*, e^{w^\dagger_{\min}}\right) \in [e^{w_{n'}}, e^{w_{n'+1}}]$ and $(a_j-\gamma^*) \in [a_{j'}, a_{j'+1}]$, and nodes that are used for linear interpolation are $(\mathbf{x}^{\hat{m}}_{n',j'}, \ldots, \mathbf{x}^{\hat{m}}_{n'+1,j'+1})$. We note that these node could be outside $\Omega_{\text{in}} \cup \Omega_{a_{\min}}$, in $\Omega_{w_{\min}} \cup \Omega_{wa_{\min}}$. However, by (5.8), the numerical solutions at these nodes satisfy the same bounds (5.6)-(5.7). Computing $v^{\hat{m}+}_{n,j}$ using linear interpolation results in

$$v^{\hat{m}+}_{n,j} = x_a\left(x_w\ v^{\hat{m}}_{n',j'} + (1-x_w)\ v^{\hat{m}}_{n'+1,j'}\right) + (1-x_a)\left(x_w\ v^{\hat{m}}_{n',j'+1} + (1-x_w)\ v^{\hat{m}}_{n'+1,j'+1}\right), \tag{5.13}$$

where $0 \le x_a \le 1$ and $0 \le x_w \le 1$ are interpolation weights. In particular,

$$x_a = \frac{a_{j'+1} - (a_j - \gamma^*)}{a_{j'+1} - a_{j'}}. \tag{5.14}$$

Using (5.8) and the induction hypothesis for (5.6) gives abound for nodal values used in (5.13)

$$\left\{v^{\hat{m}}_{n',j'}, v^{\hat{m}}_{n'+1,j'}\right\} \le e^{2\hat{m}\epsilon\frac{\Delta\tau}{T}}(\|v^0\|_\infty + a_{j'}), \quad \left\{v^{\hat{m}}_{n',j'+1},\ v^{\hat{m}}_{n'+1,j'+1}\right\} \le e^{2\hat{m}\epsilon\frac{\Delta\tau}{T}}(\|v^0\|_\infty + a_{j'+1}). \tag{5.15}$$

Taking into account the non-negative weights in linear interpolation, particularly (5.14), and upper bounds in (5.15), the interpolated result $\mathcal{I}\left\{v^{\hat{m}}\right\}(\cdot)$ in (5.12) is bounded by

$$\mathcal{I}\left\{v^{\hat{m}}\right\}\left(\max\left(e^{w_n}-\gamma^*, e^{w^\dagger_{\min}}\right), a_j-\gamma^*\right) \le e^{2\hat{m}\epsilon\frac{\Delta\tau}{T}}(\|v^0\|_\infty + (a_j-\gamma^*)). \tag{5.16}$$

Using (5.16) and $f(\gamma^*) \le \gamma^*$ (by definition in (4.15)), (5.12) becomes

$$v^{\hat{m}+}_{n,j} \le e^{2\hat{m}\epsilon\frac{\Delta\tau}{T}}\left(\|v^0\|_\infty + a_j - \gamma^*\right) + \gamma^* \le e^{2\hat{m}\epsilon\frac{\Delta\tau}{T}}\left(\|v^0\|_\infty + a_j\right),$$

which proves (5.9) at $m = \hat{m}$.

For subsequent use, we note, since $v^{\hat{m}+}_{n,j} = \max\left((v_{loc})^{\hat{m}+}_{n,j}, (v_{nlc})^{\hat{m}+}_{n,j}\right)$, (5.9) results in

$$\left\{(v_{loc})^{\hat{m}+}_{n,j}, (v_{nlc})^{\hat{m}+}_{n,j}\right\} \ \le\ v^{\hat{m}+}_{n,j} \ \le\ e^{2\hat{m}\epsilon\frac{\Delta\tau}{T}}\left(\|v^0\|_\infty + a_j\right). \tag{5.17}$$

Next, we derive a lower bound for $(v_{loc})^{\hat{m}+}_{n,j}$ and $(v_{nlc})^{\hat{m}+}_{n,j}$. By the induction hypothesis for (5.7), we have $v^{\hat{m}}_{n,j} \ge -2m\epsilon\frac{\Delta\tau}{T}e^{2\hat{m}\epsilon\frac{\Delta\tau}{T}}\left(\left\|v^0\right\|_\infty + a_j\right)$. Comparing $(v_{loc})^{\hat{m}+}_{n,j}$ given by the supremum in (4.18) with $v^{\hat{m}}_{n,j}$, which is the candidate for the supremum evaluated at $\gamma^{\hat{m}}_{n,j} = 0$, yields

$$(v_{loc})^{\hat{m}+}_{n,j} \ \ge\ v^{\hat{m}}_{n,j} \ \ge\ -2\hat{m}\epsilon\frac{\Delta\tau}{T}e^{2\hat{m}\epsilon\frac{\Delta\tau}{T}}\left(\left\|v^0\right\|_\infty + a_j\right), \tag{5.18}$$

which proves (5.10) at $m = \hat{m}$.

For $(v_{nlc})^{\hat{m}+}_{n,j}$ in (4.18), consider optimal $\gamma = \gamma^*$, where $\gamma^* \in (C_r\Delta\tau, a_j]$. Using the induction hypothesis and non-negative weights of linear interpolation, noting $\gamma^* \ge 0$ and assuming $f(\gamma^*) \ge 0$, gives

$$(v_{nlc})^{\hat{m}+}_{n,j} \ge -2\hat{m}\epsilon\frac{\Delta\tau}{T}e^{2\hat{m}\epsilon\frac{\Delta\tau}{T}}\left(\left\|v^0\right\|_\infty + (a_j-\gamma^*)\right) + f(\gamma^*) \ge -2\hat{m}\epsilon\frac{\Delta\tau}{T}e^{2\hat{m}\epsilon\frac{\Delta\tau}{T}}\left(\left\|v^0\right\|_\infty + a_j\right). \tag{5.19}$$

From (5.17)-(5.18) and (5.19), noting $\epsilon \le 1/2$, we have

$$\left\{|\,(v_{loc})^{\hat{m}+}_{n,j}\,|, |\,(v_{nlc})^{\hat{m}+}_{n,j}\,|\right\} \le e^{2\hat{m}\epsilon\frac{\Delta\tau}{T}}\left(\|v^0\|_\infty + a_j\right). \tag{5.20}$$

Step 2 - Bound for $v^{\hat{m}+1}_{n,j}$: We will show that (5.6)-(5.7) hold at $m = \hat{m}+1$. For all $n = -N/2+1, \ldots, N/2-1$, and $j = 0, \ldots, J$, we have $\left|(v_{loc})^{\hat{m}+1}_{n,j}\right| = \left|\sum_{l=-N^\dagger/2}^{N^\dagger/2-1} \tilde{g}_{n-l}\ (v_{loc})^{\hat{m}+}_{l,j}\right| \ldots$

$$\begin{aligned}\ldots \le \Delta w \sum_{l=-N^\dagger/2}^{N^\dagger/2-1} |\tilde{g}_{n-l}|\ |\,(v_{loc})^{\hat{m}+}_{l,j}\,| &\overset{\text{(i)}}{\le} e^{2\hat{m}\epsilon\frac{\Delta\tau}{T}}(\|v^0\|_\infty + a_j)\Delta w \sum_{l=-N^\dagger/2}^{N^\dagger/2-1}(\max(\tilde{g}_{n-l}, 0) + |\min(\tilde{g}_{n-l}, 0)|) \\ &\overset{\text{(ii)}}{\le} e^{2\hat{m}\epsilon\frac{\Delta\tau}{T}}(\|v^0\|_\infty + a_j)(1 + 2\epsilon\Delta\tau/T) \\ &\le e^{2(\hat{m}+1)\epsilon\frac{\Delta\tau}{T}}(\|v^0\|_\infty + a_j).\end{aligned} \tag{5.21}$$

Here, (i) comes from (5.20), and (ii) comes from (5.1). Similarly, for $n = -N/2+1, \ldots, N/2-1$, and $j = 0, \ldots, J$, we also have

$$|\,(v_{nlc})^{\hat{m}+1}_{n,j}\,| \le e^{2(\hat{m}+1)\epsilon\frac{\Delta\tau}{T}}(\|v^0\|_\infty + a_j). \tag{5.22}$$

Therefore, from (5.21)-(5.22), we conclude, for $n = -N/2+1, \ldots, N/2 - 1$, and $j = 0, \ldots, J$,

$$|v^{\hat{m}+1}_{n,j}| \le e^{2(\hat{m}+1)\epsilon\frac{\Delta\tau}{T}}(\|v^0\|_\infty + a_j),$$

which is bounded above by $e^2(\|v^0\|_\infty + z_0)$ independently of h, since $m\Delta\tau \le T$. This proves (5.6) at time $m = \hat{m}+1$.

To prove (5.7) at $m = \hat{m}+1$, note that $(v_{loc})^{\hat{m}+1}_{n,j} = \Delta w \sum_{l=-N^\dagger/2}^{N^\dagger/2-1} \tilde{g}_{n-l}\ (v_{loc})^{\hat{m}+}_{l,j} \ldots$

$$\begin{aligned}
\ldots &\ge -2\hat{m}\epsilon\frac{\Delta\tau}{T}e^{2\hat{m}\epsilon\frac{\Delta\tau}{T}}(\|v^0\|_\infty + a_j)\Delta w \sum_{l=-N^\dagger/2}^{N^\dagger/2-1} \max(\tilde{g}_{n-l},0) - e^{2\hat{m}\epsilon\frac{\Delta\tau}{T}}(\|v^0\|_\infty + a_j)\Delta w \sum_{l=-N^\dagger/2}^{N^\dagger/2-1} |\min(\tilde{g}_{n-l},0)| \\
&\ge -2\hat{m}\epsilon\frac{\Delta\tau}{T}e^{2\hat{m}\epsilon\frac{\Delta\tau}{T}}(\|v^0\|_\infty + a_j)\Delta w \sum_{l=-N^\dagger/2}^{N^\dagger/2-1} (\max(\tilde{g}_{n-l},0) + |\min(\tilde{g}_{n-l},0)|) \\
&\ge -2\hat{m}\epsilon\frac{\Delta\tau}{T}e^{2\hat{m}\epsilon\frac{\Delta\tau}{T}}(\|v^0\|_\infty + a_j)(1 + 2\epsilon\frac{\Delta\tau}{T}) \ \ge -2(\hat{m}+1)\epsilon\frac{\Delta\tau}{T}e^{2(\hat{m}+1)\epsilon\frac{\Delta\tau}{T}}\left(\|v^0\|_\infty + a_j\right).
\end{aligned}$$

This proves (5.7) at $m = \hat{m}+1$ and concludes the proof. □

Remark 5.1. *In the above proof, to derive* (5.19)*, for simplicity, we assume that, for an optimal* $\gamma^* \in (Cr\Delta\tau, a_j]$, $f(\gamma^*) \ge 0$. *If this is not the case, we still have* ℓ_∞*-stability with* (5.6) *becoming* $\left[v^m_j\right]_{\max} \le e^{2m\epsilon\frac{\Delta\tau}{T}}\left(\|v^0\|_\infty + a_j + c\right)$, *and* (5.7) *becoming* $\left[v^m_j\right]_{\min} \ge -2m\epsilon\frac{\Delta\tau}{T}e^{2m\epsilon\frac{\Delta\tau}{T}}(\|v^0\|_\infty + a_j + c)$, *and hence* (5.5) *becomes* $\left\|v^m_j\right\|_\infty \le e^{2m\epsilon\frac{\Delta\tau}{T}}\left(\|v^0\|_\infty + a_j + c\right)$, *noting the constant fixed cost* $c > 0$. *The assumption* $0 < \epsilon \le 1/2$ *is entirely for ease of exposition, and is trivially satisfied in any setting.*

Finally, if $\epsilon = 0$, *i.e. strictly monotone, the lower bounds* (5.7) *and* (5.10) *become zero, while the upper bounds* (5.6) *and* (5.9) *become* $\|v^0\|_\infty + a_j$, *which are the same as bounds established in [19] for a monotone finite difference method for fixed computational domain.*

5.2 Consistency

While equations (4.11), (4.12), (4.16), and (4.25) are convenient for computation, they are not in a form amendable for analysis. For purposes of verifying consistency, it is more convenient to rewrite them in a single equation. Unless noted otherwise, in the following, $j = 0, \ldots, J$, and $m = 0, \ldots, M-1$.

For $(w_n, a_j, \tau_{m+1}) \in \Omega_{w_{\min}} \cup \Omega_{wa_{\min}}$, i.e. $n = -N^\dagger/2, \ldots, -N/2$, we define the operators

$$\begin{aligned}
\mathcal{A}^{m+1}_{n,j}\left(h, v^{m+1}_{n,j}, \{v^m_{l,k}\}_{k\le j}\right) &= \frac{1}{\Delta\tau}\left[v^{m+1}_{n,j} - \sup_{\gamma^m_{n,j}\in[0,\min(a_j,C_r\Delta\tau)]}\left(\tilde{v}^m_{n,j} + f\left(\gamma^m_{n,j}\right)\right) + \Delta\tau\left(rv^{m+1}_{n,j}\right)\right], \\
\mathcal{B}^{m+1}_{n,j}\left(h, v^{m+1}_{n,j}, \{v^m_{l,k}\}_{k\le j}\right) &= v^{m+1}_{n,j} - \sup_{\gamma^m_{n,j}\in(C_r\Delta\tau,a_j]}\left(\tilde{v}^m_{n,j} + f\left(\gamma^m_{n,j}\right)\right) + \Delta\tau\left(rv^{m+1}_{n,j}\right),
\end{aligned} \tag{5.23}$$

where $\tilde{v}^m_{n,j}$, $n = -N^\dagger/2, \ldots, -N/2$, is given in (4.13), and $f(\cdot)$ is defined in (4.15).

For $(w_n, a_j, \tau_{m+1}) \in \Omega_{\text{in}} \cup \Omega_{a_{\min}}$, i.e. $n = -N/2+1, \ldots N/2-1$, we define the operators

$$
\begin{aligned}
\mathcal{C}_{n,j}^{m+1}\left(h, v_{n,j}^{m+1}, \{v_{l,k}^{m}\}_{k\le j}\right) \;=\; & \frac{1}{\Delta\tau}\Bigg[v_{n,j}^{m+1} \;-\Delta w \sum_{l=-N/2+1}^{N/2-1} \tilde{g}_{n-l} \sup_{\gamma_{l,j}^{m}\in[0,\min(a_j, C_r\Delta\tau)]} \left(\tilde{v}_{l,j}^{m} + f\left(\gamma_{l,j}^{m}\right)\right) \\
& \qquad - \Delta w \sum_{l=-N^{\dagger}/2}^{-N/2} \tilde{g}_{n-l}\; v_{l,j}^{m} - \Delta w \sum_{l=N/2}^{N^{\dagger}/2-1} \tilde{g}_{n-l}\; v_{l,j}^{m}\Bigg], \\
\mathcal{D}_{n,j}^{m+1}\left(h, v_{n,j}^{m+1}, \{v_{l,k}^{m}\}_{k\le j}\right) \;=\; & v_{n,j}^{m+1} \;-\Delta w \sum_{l=-N/2+1}^{N/2-1} \tilde{g}_{n-l} \sup_{\gamma_{l,j}^{m}\in(C_r\Delta\tau, a_j]} \left(\tilde{v}_{l,j}^{m} + f\left(\gamma_{l,j}^{m}\right)\right) \\
& \qquad -\Delta w \sum_{l=-N^{\dagger}/2}^{-N/2} \tilde{g}_{n-l}\; v_{l,j}^{m} - \Delta w \sum_{l=N/2}^{N^{\dagger}/2-1} \tilde{g}_{n-l}\; v_{l,j}^{m},
\end{aligned} \tag{5.24}
$$

where $\tilde{v}_{l,j}^{m}$, $l = -N/2+1, \ldots, N/2-1$, is given (4.17), and $f(\cdot)$ is defined in (4.15).

Using $\mathcal{A}_{n,j}^{m+1}(\cdot)$, $\mathcal{B}_{n,j}^{m+1}(\cdot)$, $\mathcal{C}_{n,j}^{m+1}(\cdot)$ and $\mathcal{D}_{n,j}^{m+1}(\cdot)$ defined above, our numerical scheme at the reference node $(w_n, a_j, \tau_{m+1}) \in \Omega$ can be rewritten in an equivalent form

$$
\begin{aligned}
0 \;=\; & \mathcal{H}_{n,j}^{m+1}\left(h, v_{n,j}^{m+1}, \{v_{l,k}^{m}\}_{k\le j}\right) \\
\equiv\; & \begin{cases}
\mathcal{A}_{n,j}^{m+1}(\cdot) & w_{\min}^{\dagger} \le w_n \le w_{\min}, \quad 0 \le a_j \le C_r\Delta\tau, \quad 0 < \tau_{m+1} \le T, \\
\min\left\{\mathcal{A}_{n,j}^{m+1}(\cdot), \mathcal{B}_{n,j}^{m+1}(\cdot)\right\} & w_{\min}^{\dagger} \le w_n \le w_{\min}, \quad C_r\Delta\tau < a_j \le a_J, \quad 0 < \tau_{m+1} \le T, \\
\mathcal{C}_{n,j}^{m+1}(\cdot) & w_{\min} < w_n < w_{\max}, \quad 0 \le a_j \le C_r\Delta\tau, \quad 0 < \tau_{m+1} \le T, \\
\min\left\{\mathcal{C}_{n,j}^{m+1}(\cdot), \mathcal{D}_{n,j}^{m+1}(\cdot)\right\} & w_{\min} < w_n < w_{\max}, \quad C_r\Delta\tau < a_j \le a_J, \quad 0 < \tau_{m+1} \le T, \\
v_{n,j}^{m+1} - e^{-\beta\tau_{m+1}} e^{w_n} & w_{\max} \le w_n \le w_{\max}^{\dagger}, \quad 0 \le a_j \le a_J, \quad 0 < \tau_{m+1} \le T \\
v_{n,j}^{m+1} - \max(e^{w_n}, (1-\mu)a_j - c) & w_{\min}^{\dagger} \le w_n \le w_{\max}^{\dagger}, \quad 0 \le a_j \le a_J, \quad \tau_{m+1} = 0.
\end{cases}
\end{aligned} \tag{5.25}
$$

To verify the consistency in the viscosity sense of (5.25), we first need some supporting results related to local consistency of our scheme. To this end, we define operators $F_{\text{in}'}$ and $F_{w'_{\min}}$ for the case $0 \le a_j \le C_r\Delta\tau$, i.e. $0 \le a/\Delta\tau \le C_r$,

$$
\begin{aligned}
F_{\text{in}'}(\mathbf{x}, v) \;&=\; v_\tau - \mathcal{L}v - \mathcal{J}v - \sup_{\hat{\gamma}\in[0, a/\Delta\tau]} \hat{\gamma}\left(1 - e^{-w}v_w - v_a\right)\mathbf{1}_{\{a>0\}}, \quad 0 \le a/\Delta\tau \le C_r, \\
F_{w'_{\min}}(\mathbf{x}, v) \;&=\; v_\tau + rv - \sup_{\hat{\gamma}\in[0, a/\Delta\tau]} \hat{\gamma}\left(1 - v_a\right)\mathbf{1}_{\{a>0\}}, \quad 0 \le a/\Delta\tau \le C_r.
\end{aligned} \tag{5.26}
$$

Below, we state the key supporting lemma related to local consistency of scheme (5.25).

Lemma 5.2 (Local consistency)**.** *Suppose that (i) the discretization parameter h satisfies* (4.10)*, (ii) linear interpolation in* (4.13) *and* (4.17) *is used, and (iii) $w_{\min}$ satisfies*

$$
e^{w_{\min}} - e^{w_{\min}^{\dagger}} \ge C_r\Delta\tau. \tag{5.27}
$$

Then, for any test function $\phi \in \mathcal{G}(\Omega^\infty) \cap \mathcal{C}^\infty(\Omega^\infty)$, with $\phi_{n,j}^{m} = \phi\left(\mathbf{x}_{n,j}^{m}\right)$ and $\mathbf{x} = (w_n, a_j, \tau_{m+1}) \in \Omega$, and for a sufficiently small h, we have

$$
\begin{aligned}
& \mathcal{H}_{n,j}^{m+1}\left(h, \phi_{n,j}^{m+1} + \xi, \{\phi_{l,k}^{m} + \xi\}_{k\le j}\right) \\
&= \begin{cases}
F_{in}(\cdot,\cdot) \;+ c(\mathbf{x})\,\xi + \mathcal{O}(h) + \mathcal{E}(\mathbf{x}_{n,j}^{m}, h) & w_{\min} < w_n < w_{\max}, \quad C_r\Delta\tau < a_j \le a_J, \quad 0 < \tau_{m+1} \le T; \\
F_{in'}(\cdot,\cdot) \;+ c(\mathbf{x})\,\xi + \mathcal{O}(h) + \mathcal{E}(\mathbf{x}_{n,j}^{m}, h) & w_{\min} < w_n < w_{\max}, \quad 0 < a_j \le C_r\Delta\tau, \quad 0 < \tau_{m+1} \le T; \\
F_{a_{\min}}(\cdot,\cdot) \;+ c(\mathbf{x})\,\xi + \mathcal{O}(h) & w_{\min} < w_n < w_{\max}, \quad a_j = 0, \quad 0 < \tau_{m+1} \le T; \\
F_{w_{\min}}(\cdot,\cdot) \;+ c(\mathbf{x})\,\xi + \mathcal{O}(h) & w_{\min}^{\dagger} \le w_n \le w_{\min}, \quad C_r\Delta\tau < a_j \le a_J, \quad 0 < \tau_{m+1} \le T; \\
F_{w'_{\min}}(\cdot,\cdot) \;+ c(\mathbf{x})\,\xi + \mathcal{O}(h) & w_{\min}^{\dagger} \le w_n \le w_{\min}, \quad 0 < a_j \le C_r\Delta\tau, \quad 0 < \tau_{m+1} \le T; \\
F_{wa_{\min}}(\cdot,\cdot) + c(\mathbf{x})\,\xi + \mathcal{O}(h) & w_{\min}^{\dagger} \le w_n \le w_{\min}, \quad a_j = 0, \quad 0 < \tau_{m+1} \le T; \\
F_{w_{\max}}(\cdot,\cdot) \;+ c(\mathbf{x})\,\xi & w_{\max} \le w_n \le w_{\max}^{\dagger}, \quad 0 \le a_j \le a_J, \quad 0 < \tau_{m+1} \le T; \\
F_{\tau_0}(\cdot,\cdot) \;+ c(\mathbf{x})\,\xi & w_{\min}^{\dagger} \le w_n \le w_{\max}^{\dagger}, \quad 0 \le a_j \le a_J, \quad \tau_{m+1} = 0.
\end{cases}
\end{aligned} \tag{5.28}
$$

Here, ξ is a constant and $c(\cdot)$ is a bounded function satisfying $|c(\mathbf{x})| \le \max(r,1)$ for all $\mathbf{x} \in \Omega$, and $\mathcal{E}(\mathbf{x}_{n,j}^m, h) \to 0$ as $h \to 0$. The operators $F_{in}(\cdot,\cdot)$, $F_{a_{\min}}(\cdot,\cdot)$, $F_{w_{\min}}(\cdot,\cdot)$, $F_{wa_{\min}}(\cdot,\cdot)$, $F_{w_{\max}}(\cdot,\cdot)$ and $F_{\tau_0}(\cdot,\cdot)$, defined in (3.10)-(3.15), *as well as $F_{in'}$ and $F_{w'_{\min}}$ defined in* (5.26), *are function of* $(\mathbf{x}, \phi(\mathbf{x}))$.

To prove Lemma 5.2, starting from a discrete convolution of the Green's function $g(\cdot, \Delta\tau)$ and a function $q \in \mathcal{G}(\Omega^\infty)$, we typically need to recover an associated continuous convolution (in w) and then utilize the Fourier Transform and inverse Fourier Transform. There are two cases: (i) q is not necessarily smooth, but locally bounded (as it is in $\mathcal{G}(\Omega^\infty)$), which corresponds to non-local impulses, and (ii) q is a test function in $\mathcal{G}(\Omega^\infty) \cap \mathcal{C}^\infty(\Omega^\infty)$, which corresponds to local impulses. We first present some auxiliary results, namely Lemma 5.3 (for case (i)) and in Lemma 5.4 (for case (ii)).

Lemma 5.3 (Function in $\mathcal{G}(\Omega^\infty)$)**.** *Suppose the discretization parameter h satisfies* (4.10)*. Let $p(w,a,\tau)$ be in $\mathcal{G}(\Omega^\infty)$. For any $\mathbf{x}_{n,j}^m$, $n \in \{-N/2+1, \ldots N/2-1\}$, $j \in \{0,\ldots,J\}$ and $m \in \{1,\ldots,M\}$, we have*

$$\Delta w \sum_{l=-N^\dagger/2}^{N^\dagger/2-1} \tilde{g}_{n-l}\; p_{l,j}^m = p_{n,j}^m + \mathcal{O}(h^2) + \mathcal{E}(\mathbf{x}_{n,j}^m, h), \qquad \textit{where } \mathcal{E}(\mathbf{x}_{n,j}^m, h) \to 0 \textit{ as } h \to 0.$$

Proof of Lemma 5.3. We fix $a = a_j$ and $\tau = \tau_m$, and instead of writing $p(w, a_j, \tau_m)$, we will write $p(w)$ which is a bounded function of $w \in \mathbb{R}$. We will also write p_l instead of $p_{l,j}^m$.

Since $p(w)$ does not need to be in $L^1(\mathbb{R})$, we first construct a function $\hat{p}(w) : \mathbb{R} \to \mathbb{R}$ which is in $L^1(\mathbb{R})$ and bounded in $\mathbb{R}$ and agrees with $p(w)$ in $[w_{\min}^\dagger, w_{\max}^\dagger]$. This can be achieved by using a standard smooth cut-off function [48]. To this end, with $\hat{w}_0 = \left(w_{\min}^\dagger + w_{\max}^\dagger\right)/2$, we define $\overline{\mathbb{D}}_d(\hat{w}_0) := \{w \in \mathbb{R} : |w - \hat{w}_0| \le d\}$, the closed ball centered at $\hat{w}_0$ with radius d sufficiently large so that $[w_{\min}^\dagger, w_{\max}^\dagger]$ is contained in $\overline{\mathbb{D}}_d(\hat{w}_0)$. Consider a smooth cut-off function $\zeta(w)$, $w \in \mathbb{R}$, satisfying $0 \le \zeta(w) \le 1$, $\zeta(w) = 1$ on $\overline{\mathbb{D}}_d(\hat{w}_0)$ and $\zeta(w) = 0$ outside of $\mathbb{D}_{2d}(\hat{w}_0)$. Then the function $\hat{p}(w) = \zeta(w)\phi(w)$ satisfies our requirements.

Consider function $q : \mathbb{R} \to \mathbb{R}$ defined as follows: (i) $q(w) = \sum_{l=-N^\dagger/2}^{N^\dagger/2-1} p_l \varphi_l(w)$, $w \in [w_{\min}^\dagger, w_{\max}^\dagger]$, and (ii) $q(w) = \hat{p}(w)$, $w \in \mathbb{R} \setminus [w_{\min}^\dagger, w_{\max}^\dagger]$, where $\{\varphi_l(w)\}$ are piecewise linear basis functions given in (4.21). It is straightforward to see that $q(w)$ is in $L^1(\mathbb{R})$ and bounded in $\mathbb{R}$. We have

$$\begin{aligned}
\Delta w \sum_{l=-N^\dagger/2}^{N^\dagger/2-1} \tilde{g}_{n-l}\; p_l &\overset{\text{(i)}}{=} \Delta w \sum_{l=-N^\dagger/2}^{N^\dagger/2-1} \tilde{g}_{n-l}(\infty)\; p_l + \mathcal{E}_f \overset{\text{(ii)}}{=} \int_{w_{\min}^\dagger}^{w_{\max}^\dagger} q(w)\; \hat{g}(w_n - w, \Delta\tau)\; dw + \mathcal{E}_f + \mathcal{E}_o \\
&\overset{\text{(iii)}}{=} \int_{w_{\min}^\dagger}^{w_{\max}^\dagger} q(w)\; g(w_n - w, \Delta\tau)\; dw + \mathcal{E}_f + \mathcal{E}_o + \mathcal{E}_{\hat{g}} \\
&\overset{\text{(iv)}}{=} \int_{-\infty}^{\infty} q(w)\; g(w_n - w, \Delta\tau)\; dw + \mathcal{E}_f + \mathcal{E}_o + \mathcal{E}_{\hat{g}} + \mathcal{E}_b \\
&\overset{\text{(v)}}{=} p_n + \mathcal{E}_f + \mathcal{E}_o + \mathcal{E}_{\hat{g}} + \mathcal{E}_b + \mathcal{E}_c,
\end{aligned} \tag{5.29}$$

where the errors $\mathcal{E}_f$, $\mathcal{E}_o$, $\mathcal{E}_{\hat{g}}$, $\mathcal{E}_b$, and $\mathcal{E}_c$ are described below.

- In (i), $\mathcal{E}_f \equiv \mathcal{E}_f(\mathbf{x}_{n,j}^m, h)$ is the Fourier series error arising from truncating $\tilde{g}_{n-l}(\infty)$, defined in (4.30), to $\tilde{g}_{n-l}(\alpha)$, $\alpha \in \{2,4,8,\ldots\}$, in (4.31). As noted in (4.32), $\mathcal{E}_f(\mathbf{x}_{n,j}^m, h) = \mathcal{O}(e^{-\frac{1}{h}})$.

- In (ii), $\mathcal{E}_o \equiv \mathcal{E}_o(\mathbf{x}_{n,j}^m, h)$ is the error associated with projecting $q(w)$ onto $\varphi_l(\cdot)$, and is given by

$$\mathcal{E}_o \equiv \mathcal{E}_o(\mathbf{x}_{n,j}^m, h) = \int_{w_{\min}^\dagger}^{w_{\max}^\dagger} \left[\sum_{l=-N^\dagger/2}^{N^\dagger/2-1} p_l \varphi_l(w) - q(w)\right] \hat{g}\left(w_n - w, \Delta\tau\right)\; dw, \tag{5.30}$$

  which, by the definition of function $q(w)$, is zero.

- In (iii), the error $\mathcal{E}_{\hat{g}} \equiv \mathcal{E}_{\hat{g}}(\mathbf{x}_{n,j}^m, h)$ is due to approximating $g(w, \Delta)$ by its localized, periodic approximation $\hat{g}(w, \Delta)$, and is defined by

$$\mathcal{E}_{\hat{g}} \equiv \mathcal{E}_{\hat{g}}(\mathbf{x}_{n,j}^m, h) = \int_{w_{\min}^\dagger}^{w_{\max}^\dagger} q(w) \ \left(\hat{g}(w_n - w, \Delta\tau) - g(w_n - w, \Delta\tau)\right) \ dw. \tag{5.31}$$

  Using (4.29) with $q(w) \in L^1(\mathbb{R})$ and its boundedness in $\mathbb{R}$, we obtain $\mathcal{E}_{\hat{g}}(\mathbf{x}_{n,j}^m, h) = \mathcal{O}\left(h^2\right)$ as $h \to 0$.

- In (iv), $\mathcal{E}_b \equiv \mathcal{E}_b(\mathbf{x}_{n,j}^m, h)$ is the boundary truncation error defined in (4.5), satisfying $|\mathcal{E}_b| < K_1 \Delta\tau e^{-K_2 P^\dagger}$, where K_1 and K_2 are positive constants independent of h, hence $\mathcal{E}_b(\mathbf{x}_{n,j}^m, h) = \mathcal{O}(he^{-\frac{1}{h}})$ as $h \to 0$.

- In (v), $\mathcal{E}_c \equiv \mathcal{E}_c(\mathbf{x}_{n,j}^m, h) = \int_{-\infty}^{\infty} g(w_n - w, \Delta\tau)\left(q(w) - q(w_n)\right) \ dw$. By the "cancelation properties" of the Green's function [30, 36]), noting the continuity of $q(\cdot)$, we have $\mathcal{E}_c(\mathbf{x}_{n,j}^m, h) \ \to \ 0$ as $h \to 0$.

Letting $\mathcal{E}(\mathbf{x}_{n,j}^m, h) = \mathcal{E}_c(\mathbf{x}_{n,j}^m, h)$ concludes the proof. □

For a test function $\phi \in \mathcal{G}(\Omega^\infty) \cap \mathcal{C}^\infty(\Omega^\infty)$, we have the lemma below.

Lemma 5.4 (Test function in $\mathcal{G}(\Omega^\infty) \cap \mathcal{C}^\infty(\Omega^\infty)$)**.** *Let $\phi \in \mathcal{G}(\Omega^\infty) \cap \mathcal{C}^\infty(\Omega^\infty)$. For any $\mathbf{x}_{n,j}^m$, $n \in \{-N/2 + 1, \ldots N/2 - 1\}$, $j \in \{0, \ldots, J\}$ and $m \in \{1, \ldots, M\}$,*

$$\Delta w \sum_{l=-N^\dagger/2}^{N^\dagger/2-1} \tilde{g}_{n-l} \ \phi_{l,j}^m \ = \ \phi_{n,j}^m + \Delta\tau \left[\mathcal{L}\phi + \mathcal{J}\phi\right]_{n,j}^m + \mathcal{O}\left(h^2\right), \tag{5.32}$$

where the operators $\mathcal{L}$ and $\mathcal{J}$ are defined in (3.4)*.*

Proof of Lemma 5.4. Since we apply the Fourier transform and inverse Fourier transform with respect to w, we fix $a = a_j$ and $\tau = \tau_m$. Instead of $\phi(w, a_j, \tau_m)$, we will write $\phi(w)$, which is a smooth univariate function of $w \in \mathbb{R}$. Since $\phi(w)$ does not need to be in $L^1(\mathbb{R})$, we apply a similar smooth cut-off function as in Lemma 5.3 to obtain a smooth function $\chi(w)$ that is in $L^1(\mathbb{R})$, bounded in $\mathbb{R}$, and agrees with $\phi(w)$ in $[w_{\min}^\dagger, w_{\max}^\dagger]$. With this in mind, starting from the left-hand-side of (5.32), we apply steps (i)-(iv) in (5.29), noting that the projection error $\mathcal{E}_o(\mathbf{x}_{n,j}^m, h)$ associated with the smooth function $\chi(w)$ becomes (also noting $\chi(w_l) = \phi_{l,j}^m$)

$$\mathcal{E}_o(\mathbf{x}_{n,j}^m, h) = \int_{w_{\min}^\dagger}^{w_{\max}^\dagger} \left[\sum_{l=-N^\dagger/2}^{N^\dagger/2-1} \chi(w_l)\varphi_l(w) - \chi(w)\right] \hat{g}\left(w_n - w, \Delta\tau\right) \ dw = \mathcal{O}(h^2).$$

Here, we used Taylor series expansions and the form of $\varphi_l(w)$ given in (4.21). This gives

$$\begin{aligned} \Delta w \sum_{l=-N^\dagger/2}^{N^\dagger/2-1} \tilde{g}_{n-l} \ \chi_{l,j}^m \ &= \ \int_{-\infty}^{\infty} \chi(w) \ g\left(w_n - w, \Delta\tau\right) \ dw + \mathcal{O}(h^2) \\ &= \ [\chi * g](w_n) + \mathcal{O}(h^2) = \mathcal{F}^{-1}\left[\mathcal{F}\left[\chi\right](\eta) \ G\left(\eta, \Delta\tau\right)\right](w_n) + \mathcal{O}(h^2), \end{aligned} \tag{5.33}$$

where $[\chi * g]$ denotes the convolution of $\chi(w)$ and $g(w, \Delta\tau)$. In (5.33), with $\Psi(\eta)$ given in (4.27), expanding $G(\eta, \Delta\tau) = e^{\Psi(\eta)\Delta\tau}$ by a Taylor series gives

$$\begin{aligned} \left[\chi * g\right](w_n) \ &= \ \mathcal{F}^{-1}\left[\mathcal{F}\left[\chi\right](\eta) \ \left(1 + \Psi(\eta)\Delta\tau + R(\eta)\Delta\tau^2\right)\right](w_n) \\ &= \ \chi(w_n) + \Delta\tau\mathcal{F}^{-1}\left[\mathcal{F}\left[\chi\right](\eta) \ \Psi\left(\eta\right)\right](w_n) + \Delta\tau^2\mathcal{F}^{-1}\left[\mathcal{F}\left[\chi\right](\eta) \ R\left(\eta\right)\right](w_n), \end{aligned} \tag{5.34}$$

where $R(\eta) = \frac{1}{2}\Psi(\eta)^2 e^{\Psi(\eta)\xi}$, $\xi \in (0, \Delta\tau)$, is the remainder.
For the second term $\Delta\tau\mathcal{F}^{-1}\left[\cdot\right](w_n)$ in (5.34), first, using the closed-form expression for $\Psi(\eta)$ in (4.27) gives

$$\begin{aligned} \mathcal{F}[\chi](\eta) \ \Psi(\eta) \ &= \ \mathcal{F}\left[-\frac{\sigma^2}{2}\chi_{ww} + \left(r - \lambda\kappa - \frac{\sigma^2}{2} - \beta\right)\chi_w - (r + \lambda)\,\chi + \lambda\int_{-\infty}^{\infty} \chi(w + y) \ b(y) \ dy\right](\eta) \\ &= \mathcal{F}\left[\mathcal{L}\chi + \mathcal{J}\chi\right](\eta). \end{aligned} \tag{5.35}$$

Then, substituting (5.35) into the second term $\Delta\tau\mathcal{F}^{-1}[\cdot](w_n)$ in (5.34) gives

$$\Delta\tau\mathcal{F}^{-1}\left[\mathcal{F}[\chi](\eta)\ \Psi(\eta)\right](w_n) = \Delta\tau\left[\mathcal{L}\chi + \mathcal{J}\chi\right]_{n,j}^{m}. \tag{5.36}$$

For the third term $\Delta\tau^2\mathcal{F}^{-1}[\cdot](w_n)$ in (5.34), we have

$$\begin{aligned}
\Delta\tau^2\left|\mathcal{F}^{-1}\left[\mathcal{F}[\chi](\eta)\ R(\eta)\right](w_n)\right| &= \Delta\tau^2\left|\int_{-\infty}^{\infty} e^{2\pi i\eta w_n} R(\eta)\left[\int_{-\infty}^{\infty} e^{-2\pi i\eta w}\chi(w)\ dw\right]d\eta\right| \\
&\leq \Delta\tau^2\int_{-\infty}^{\infty}|\chi(w)|\ dw\ \int_{-\infty}^{\infty}|R(\eta)|\ d\eta \\
&\overset{\text{(i)}}{=} \Delta\tau^2\int_{-\infty}^{\infty}|\chi(w)|\ dw\ \int_{-\infty}^{\infty}\frac{1}{2}\left|\Psi(\eta)\right|^2\ e^{\mathrm{Re}(\Psi(\eta))\xi}\ d\eta \\
&\overset{\text{(ii)}}{\leq} \Delta\tau^2\int_{-\infty}^{\infty}|\chi(w)|\ dw\ \int_{-\infty}^{\infty}\frac{1}{2}\left|\Psi(\eta)\right|^2\ e^{-\frac{1}{2}\xi\sigma^2(2\pi\eta)^2}\ d\eta \\
&\overset{\text{(iii)}}{=} \mathcal{O}(\Delta\tau^2).
\end{aligned} \tag{5.37}$$

Here, in (i), we use $R(\eta) = \frac{1}{2}\Psi(\eta)^2 e^{\Psi(\eta)\xi}$ and $\mathrm{Re}(\Psi(\eta))$ is the real part of $\Psi(\eta)$. In (ii), using the closed-form expression of $\Psi(\eta)$ in (4.27) and noting that $\mathrm{Re}(\overline{B}(\eta)) \leq 1$ and $r > 0$, we have

$$\mathrm{Re}(\Psi(w)) = -\frac{1}{2}\sigma^2(2\pi\eta)^2 - (r+\lambda) + \lambda\mathrm{Re}(\overline{B}(\eta)) \leq -\frac{1}{2}\sigma^2(2\pi\eta)^2.$$

In (iii), we note $\chi(w) \in L^1(\mathbb{R})$, and the second integral is bounded by a constant, since $|\Psi(\eta)|^2$ is a quartic polynomial in η, and $\int_{-\infty}^{\infty}|\eta|^k e^{-\frac{1}{2}\xi\sigma^2(2\pi\eta)^2}d\eta$, $k \in \{0,1,2,3,4\}$, are bounded. Substituting (5.36) and (5.37) back into (5.34), noting (5.33) and the definition of $\chi(w)$, gives

$$\Delta w \sum_{l=-N^\dagger/2}^{N^\dagger/2-1} \tilde{g}_{n-l}\ \phi_{l,j}^{m} \;=\; \phi_{n,j}^{m} + \Delta\tau\left[\mathcal{L}\phi + \mathcal{J}\phi\right]_{n,j}^{m} + \mathcal{O}(h^2). \tag{5.38}$$

□

We are now ready to present a proof of Lemma 5.2.

Proof of Lemma 5.2. Since $\phi \in \mathcal{C}^\infty(\Omega^\infty)$ and Ω is bounded, ϕ has continuous and bounded derivatives of up to second-order in Ω. We now show that the first equation of (5.28) is true, that is,

$$\begin{aligned}
\mathcal{H}_{n,j}^{m+1}(\cdot) = \min\left\{\mathcal{C}_{n,j}^{m+1}(\cdot), \mathcal{D}_{n,j}^{m+1}(\cdot)\right\} &= F_{\text{in}}(\mathbf{x}, \phi(\mathbf{x})) + c(\mathbf{x})\xi + \mathcal{O}(h) + \mathcal{E}(\mathbf{x}_{n,j}^{m}, h) \\
&\text{if } w_{\min} < w_n < w_{\max},\ C_r\Delta\tau < a_j \leq a_J,\ 0 < \tau_{m+1} \leq T,
\end{aligned}$$

where operators $\mathcal{C}_{n,j}^{m+1}(\cdot)$ and $\mathcal{D}_{n,j}^{m+1}(\cdot)$ are defined in (5.24). In this case, operator $\mathcal{C}_{n,j}^{m+1}(\cdot)$ is written as

$$\begin{aligned}
\mathcal{C}_{n,j}^{m+1}(\cdot) = \frac{1}{\Delta\tau}\Bigg[\phi_{n,j}^{m+1} + \xi - \Delta w\sum_{l=-N/2+1}^{N/2-1}\tilde{g}_{n-l}\sup_{\gamma_{l,j}^{m}\in[0,C_r\Delta\tau]}\left(\tilde{\phi}_{l,j}^{m} + f\left(\gamma_{l,j}^{m}\right)\right) \\
- \Delta w\sum_{l=-N^\dagger/2}^{-N/2}\tilde{g}_{n-l}\ \left(\phi_{l,j}^{m} + \xi\right) - \Delta w\sum_{l=N/2}^{N^\dagger/2-1}\tilde{g}_{n-l}\ \left(\phi_{l,j}^{m} + \xi\right)\Bigg],
\end{aligned} \tag{5.39}$$

$$\text{where}\quad \tilde{\phi}_{l,j}^{m} + f\left(\gamma_{l,j}^{m}\right) = \mathcal{I}\left\{\phi\left(\mathbf{x}^{m}\right) + \xi\right\}\left(\ln\left(\max\left(e^{w_l} - \gamma_{l,j}^{m}, e^{w_{\min}^\dagger}\right)\right), a_j - \gamma_{l,j}^{m}\right) + \gamma_{l,j}^{m}. \tag{5.40}$$

Condition (5.27) implies that, for any $w_l \in (w_{\min}, w_{\max})$, $e^{w_l} - \gamma_{l,j}^{m} > e^{w_{\min}^\dagger}$ for all $\gamma_{l,j}^{m} \in [0, C_r\Delta\tau]$, and hence, we can eliminate the $\max(\cdot)$ operator in the linear interpolation operator in (5.40) when $\gamma_{l,j}^{m} \in [0, C_r\Delta\tau]$. Consequently, with $\gamma_{l,j}^{m} \in [0, C_r\Delta\tau]$, (5.40) becomes

$$\begin{aligned}
\tilde{\phi}_{l,j}^{m} + f\left(\gamma_{l,j}^{m}\right) &\overset{\text{(i)}}{=} \phi\left(\ln\left(e^{w_l} - \gamma_{l,j}^{m}\right), a_j - \gamma_{l,j}^{m}, \tau_m\right) + \xi + \mathcal{O}\left(\left(\Delta w + \Delta a_{\max}\right)^2\right) + \gamma_{l,j}^{m} \\
&\overset{\text{(ii)}}{=} \phi_{l,j}^{m} + \xi + \gamma_{l,j}^{m}\left(1 - e^{-w_l}(\phi_w)_{l,j}^{m} - (\phi_a)_{l,j}^{m}\right) + \mathcal{O}\left(h^2\right).
\end{aligned} \tag{5.41}$$

Here, in (i), due to linear interpolation, we obtain an error of size $\mathcal{O}\left((\Delta w + \Delta a_{\max})^2\right)$, and also we can completely separate ξ from interpolated values; and in (ii), we apply a Taylor series to expand $\phi\left(\ln\left(e^{w_l} - \gamma_{l,j}^m\right), a_j - \gamma_{l,j}^m, \tau_m\right)$ about (w_l, a_j, τ_m), noting $\gamma_{l,j}^m = \mathcal{O}(\Delta\tau)$.

In (5.41), since the control $\gamma_{l,j}^m$ can be factored out completely from the objective function, namely $\gamma_{l,j}^m\left(1 - e^{-w_l}(\phi_w)_{l,j}^m - (\phi_a)_{l,j}^m\right)$, we define a new control variable $\hat{\gamma}_{l,j}^m = \gamma_{l,j}^m/\Delta\tau \in [0, C_r]$. With this in mind, let $\phi'(\hat{\gamma}, \mathbf{x}')$ be a function of $\hat{\gamma} \in [0, C_r]$ and $\mathbf{x}' = (w', a', \tau') \in \Omega^\infty$ defined by

$$\phi'(\hat{\gamma}, \mathbf{x}') = \begin{cases} \hat{\gamma}\left(1 - e^{-w}\phi_w(\mathbf{x}') - \phi_a(\mathbf{x}')\right), & w_{\min} < w' < w_{\max},\ C_r\Delta\tau < a' \le a_J,\ 0 \le \tau' < T, \\ 0 & \text{otherwise.} \end{cases} \tag{5.42}$$

Using (5.42), operator $\mathcal{C}_{n,j}^m(\cdot)$ in (5.39) can be written as

$$\begin{aligned} \mathcal{C}_{n,j}^{m+1}(\cdot) &= \frac{1}{\Delta\tau}\left[\phi_{n,j}^{m+1} - \Delta w \sum_{l=-N^\dagger/2}^{N^\dagger/2-1} \tilde{g}_{n-l}\ \phi_{l,j}^m + \xi\left(1 - \Delta w \sum_{l=-N^\dagger/2}^{N^\dagger/2-1} \tilde{g}_{n-l}\right) + \mathcal{O}\left(h^2\right)\right] \\ &\qquad - \Delta w \sum_{l=-N^\dagger/2}^{N^\dagger/2-1} \tilde{g}_{n-l} \sup_{\hat{\gamma}\in[0,C_r]} \phi'\left(\hat{\gamma}, \mathbf{x}_{l,j}^m\right). \end{aligned} \tag{5.43}$$

For the term $\Delta w \sum_l \tilde{g}_{n-l}\ \phi_{l,j}^m$ in (5.43), using Lemma 5.4 on the smooth function $\phi(\cdot)$ at $\mathbf{x}_{n,j}^m$ gives

$$\Delta w \sum_{l=-N^\dagger/2}^{N^\dagger/2-1} \tilde{g}_{n-l}\ \phi_{l,j}^m = \phi_{n,j}^m + \Delta\tau\left[\mathcal{L}\phi + \mathcal{J}\phi\right]_{n,j}^m + \mathcal{O}\left(h^2\right). \tag{5.44}$$

Regarding $\Delta w \sum_{l=-N/2+1}^{N/2-1} \tilde{g}_{n-l} \sup_{\hat{\gamma}\in[0,C_r]} \phi'(\cdot)$ in (5.43), note that $\sup_{\hat{\gamma}\in[0,C_r]} \phi'(\hat{\gamma}, \mathbf{x}')$ is a function of $\mathbf{x}'$, and is in $\mathcal{G}(\Omega^\infty)$. Applying Lemma 5.3 on $\left\{\mathbf{x}_{l,j}^m, \sup_{\hat{\gamma}\in[0,C_r]} \phi'\left(\hat{\gamma}, \mathbf{x}_{l,j}^m\right)\right\}$, $l = -N^\dagger/2, \ldots, N^\dagger/2 - 1$, gives

$$\Delta w \sum_{l=-N^\dagger/2}^{N^\dagger/2-1} \tilde{g}_{n-l} \sup_{\hat{\gamma}\in[0,C_r]} \phi'\left(\hat{\gamma}, \mathbf{x}_{l,j}^m\right) = \left[\sup_{\hat{\gamma}\in[0,C_r]} \hat{\gamma}\left(1 - e^{-w}\phi_w - \phi_a\right)\right]_{n,j}^m + \mathcal{O}(h^2) + \mathcal{E}(\mathbf{x}_{n,j}^m, h), \tag{5.45}$$

where $\mathcal{E}(\mathbf{x}_{n,j}^m, h) \to 0$ as $h \to 0$. Also, in (5.43), the term $\Delta w \sum_{l=-N^\dagger/2}^{N^\dagger/2-1} \tilde{g}_{n-l} = e^{-r\Delta\tau}$ by (5.1). Substituting this result and (5.44)-(5.45) into (5.43) gives

$$\begin{aligned} \mathcal{C}_{n,j}^{m+1}(\cdot) &\overset{\text{(i)}}{=} \frac{\phi_{n,j}^{m+1} - \phi_{n,j}^m}{\Delta\tau} - \left[\mathcal{L}\phi + \mathcal{J}\phi\right]_{n,j}^m + \left[\sup_{\hat{\gamma}\in[0,C_r]} \hat{\gamma}\left(1 - e^{-w}\phi_w - \phi_a\right)\right]_{n,j}^m + r\xi + \mathcal{O}(h) + \mathcal{E}(\mathbf{x}_{n,j}^m, h) \\ &\overset{\text{(ii)}}{=} \left[\phi_\tau - \mathcal{L}\phi - \mathcal{J}\phi - \sup_{\hat{\gamma}\in[0,C_r]} \hat{\gamma}\left(1 - e^{-w}\phi_w - \phi_a\right)\right]_{n,j}^{m+1} + r\xi + \mathcal{O}(h) + \mathcal{E}(\mathbf{x}_{n,j}^m, h). \end{aligned}$$

Here, in (i) we have $\frac{\xi}{\Delta\tau}\left(1 - \Delta w \sum_{l=-N^\dagger/2}^{N^\dagger/2-1} \tilde{g}_{n-l}\right) = r\xi + \mathcal{O}(h)$. In (ii), we use

$$(\phi_\tau)_{n,j}^m = (\phi_\tau)_{n,j}^{m+1} + \mathcal{O}(h),\ (\phi_w)_{n,j}^m = (\phi_w)_{n,j}^{m+1} + \mathcal{O}(h),\ (\phi_a)_{n,j}^m = (\phi_a)_{n,j}^{m+1} + \mathcal{O}(h).$$

This step results in an $\mathcal{O}(h)$ term inside $\sup_{\hat{\gamma}}(\cdot)$, which can be moved out of the $\sup_{\hat{\gamma}}(\cdot)$, because it has the form $K(\hat{\gamma})h$, where $K(\hat{\gamma})$ is bounded independently of h, due to boundedness of $\hat{\gamma} \in [0, C_r]$ independently of h.

For operator $\mathcal{D}_{n,j}^{m+1}(\cdot)$, we have

$$\begin{aligned} \mathcal{D}_{n,j}^{m+1}(\cdot) &= \left(\phi_{n,j}^{m+1} + \xi\right) - \Delta w \sum_{l=-N/2+1}^{N/2-1} \tilde{g}_{n-l} \sup_{\gamma_{l,j}^m \in (C_r\Delta\tau, a_j]} \left(\tilde{\phi}_{l,j}^m + f\left(\gamma_{l,j}^m\right)\right) \\ &\qquad - \Delta w \sum_{l=-N^\dagger/2}^{-N/2} \tilde{g}_{n-l}\ \left(\phi_{l,j}^m + \xi\right) - \Delta w \sum_{l=N/2}^{N^\dagger/2-1} \tilde{g}_{n-l}\ \left(\phi_{l,j}^m + \xi\right), \end{aligned} \tag{5.46}$$

$$\begin{aligned} \text{where}\ \ \tilde{\phi}_{l,j}^m + f\left(\gamma_{l,j}^m\right) &= \mathcal{I}\left\{\phi\left(\mathbf{x}^m\right) + \xi\right\}\left(\ln\left(\max\left(e^{w_l} - \gamma_{l,j}^m, e^{w_{\min}^\dagger}\right)\right), a_j - \gamma_{l,j}^m\right) \\ &\quad + \gamma_{l,j}^m(1 - \mu) + \mu C r \Delta\tau - c. \end{aligned} \tag{5.47}$$

Since $\gamma_{l,j}^m \in (C_r\Delta\tau, a_j]$, we cannot eliminate the $\max(\cdot)$ operator in linear interpolation in (5.47), hence

$$\mathcal{I}\left\{\phi\left(\mathbf{x}^m\right)+\xi\right\}(\cdot) = \phi\left(\ln\left(\max\left(e^{w_l}-\gamma_{l,j}^m\right), e^{w_{\min}^\dagger}\right), a_j-\gamma_{l,j}^m, \tau_m\right) + \xi + \mathcal{O}(h^2).$$

Let $\phi''(\gamma, \mathbf{x}')$ be a function of $\gamma \in [0,a]$ and $\mathbf{x}' = (w', a', \tau') \in \Omega^\infty$ defined by

$$\phi''(\gamma,\mathbf{x}') = \begin{cases} \mathcal{M}(\gamma)\phi(\mathbf{x}') + \mu C_r \Delta\tau & w_{\min} < w' < w_{\max},\ C_r\Delta\tau < a' \leq a_J,\ 0 \leq \tau' < T, \quad (5.48\text{a}) \\ \phi(\mathbf{x}') & \text{otherwise}, \quad (5.48\text{b}) \end{cases}$$

where $\mathcal{M}(\cdot)$ is defined in (3.8b). It is straightforward to show that, for a fixed $\mathbf{x}' \in \Omega$ satisfies (5.48a), $\phi''(\gamma;\mathbf{x}')$ is (uniformly) continuous in $\gamma \in [0,a]$. Hence, for the case (5.48a)

$$\sup_{\gamma\in(C_r\Delta\tau,a']} \phi''\left(\gamma,\mathbf{x}'\right) - \sup_{\gamma\in(0,a']} \phi''\left(\gamma,\mathbf{x}'\right) = \max_{\gamma\in[C_r\Delta\tau,a']} \phi''\left(\gamma,\mathbf{x}'\right) - \max_{\gamma\in[0,a']} \phi''\left(\gamma,\mathbf{x}'\right) = \mathcal{O}(h), \quad (5.49)$$

since the difference of the optimal values of γ for the two $\max(\cdot)$ expressions is bounded by $C_r\Delta\tau = \mathcal{O}(h)$. Using (5.48), with (5.49) in mind, operator $\mathcal{D}_{n,j}^m(\cdot)$ in (5.46) can be written as

$$\mathcal{D}_{n,j}^{m+1}(\cdot) = \phi_{n,j}^{m+1} + \xi\left(1 - \Delta w \sum_{l=-N^\dagger/2}^{N^\dagger/2-1} \tilde{g}_{n-l}\right) - \Delta w \sum_{l=-N^\dagger/2}^{N^\dagger/2-1} \tilde{g}_{n-l} \sup_{\gamma\in[0,a_j]} \phi''\left(\gamma,\mathbf{x}_{l,j}^m\right) + \mathcal{O}(h). \quad (5.50)$$

Note that $\sup_{\gamma\in[0,a_j]}\phi''(\gamma,\mathbf{x}')$ is a function of $\mathbf{x}'$, and it is straightforward to show that it is in $\mathcal{G}(\Omega^\infty)$. Applying Lemma 5.3 to $\left\{\mathbf{x}_{l,j}^m, \sup_{\gamma\in[0,a]}\left(\phi''\left(\gamma,\mathbf{x}_{l,j}^m\right)\right)\right\}$, $l = -N^\dagger/2, \ldots, N^\dagger/2-1$, we obtain

$$\begin{aligned} \Delta w \sum_{l=-N^\dagger/2}^{N^\dagger/2-1} \tilde{g}_{n-l} \sup_{\gamma\in[0,a_j]} \phi''\left(\gamma,\mathbf{x}_{l,j}^m\right) &\overset{\text{(i)}}{=} \sup_{\gamma\in[0,a_j]} \mathcal{M}(\gamma)\phi\left(\mathbf{x}_{n,j}^m\right) + \mu C_r\Delta\tau + \mathcal{O}(h^2) + \mathcal{E}(\mathbf{x}_{n,j}^m, h) \\ &\overset{\text{(ii)}}{=} \sup_{\gamma\in[0,a_j]} \mathcal{M}(\gamma)\phi\left(\mathbf{x}_{n,j}^{m+1}\right) + \mathcal{O}(h) + \mathcal{E}(\mathbf{x}_{n,j}^m, h). \end{aligned}$$

Here, in (i) the error term $\mathcal{E}(\mathbf{x}_{n,j}^m, h) \to 0$ as $h \to 0$, and we use the definition (5.48a) of $\phi''(\cdot)$, and in (ii) we have $\mathcal{M}(\gamma)\phi\left(\mathbf{x}_{n,j}^m\right) = \mathcal{M}(\gamma)\phi\left(\mathbf{x}_{n,j}^{m+1}\right) + \mathcal{O}(h)$, which is combined with $\mu C_r\Delta\tau = \mathcal{O}(h)$. Substituting (5.51) into (5.50) gives

$$\mathcal{D}_{n,j}^{m+1}(\cdot) = \phi_{n,j}^{m+1} - \sup_{\gamma\in[0,a]} \mathcal{M}(\gamma)\phi\left(\mathbf{x}_{n,j}^{m+1}\right) + \mathcal{O}(h) + \mathcal{E}(\mathbf{x}_{n,j}^m, h). \quad (5.51)$$

Overall, recalling $\mathbf{x} = \mathbf{x}_{n,j}^{m+1}$, we have

$$\begin{aligned} &\mathcal{H}_{n,j}^{m+1}\left(h, \phi_{n,j}^{m+1}+\xi, \left\{\phi_{l,k}^m+\xi\right\}_{k\leq j},\right) - F_{\text{in}}\left(\mathbf{x}, \phi(\mathbf{x}), D\phi(\mathbf{x}), D^2\phi(\mathbf{x}), \mathcal{J}\phi(\mathbf{x}), \mathcal{M}\phi(\mathbf{x})\right) \\ &\quad = c(\mathbf{x})\xi + \mathcal{O}(h) + \mathcal{E}(\mathbf{x}_{n,j}^m, h), \quad \text{if } w_{\min} < w_n < w_{\max},\ C_r\Delta\tau < a_j \leq a_J,\ 0 < \tau_{m+1} \leq T, \end{aligned}$$

where $c(\cdot)$ is a bounded function satisfying $0 \leq c(\mathbf{x}) \leq r$ and $\mathcal{E}(\mathbf{x}_{n,j}^m, h) \to 0$ as $h \to 0$. This proves the first equation in (5.28). The remaining equations in (5.28) can be proved using similar arguments with the first equation. □

Remark 5.2. *We emphasize that for the limiting case $P^\dagger = \infty$ (i.e. $\Delta\tau = 0$), the Green's function $g(w,\Delta\tau)$ trivially becomes the Dirac delta function. Thus, for this case, we do not need to use the smooth cut-off function and the Fourier Transform as in Lemma 5.4. The results in Lemma 5.2, Lemma 5.3 and Lemma 5.4 are still valid for this limiting case.*

Remark 5.3. *We impose the condition (5.27) to ease the presentation of the proof, i.e. $\max(\cdot)$ in the operator $\mathcal{C}_{n,j}^{m+1}(\cdot)$ can be removed. However, we can avoid this condition by the following steps: if it is not satisfied, we find $w'_{\min}$ satisfying $e^{w'_{\min}} - e^{w^\dagger_{\min}} \geq C_r\Delta\tau$. For the range $w \in [w^\dagger_{\min}, w'_{\min}]$, we employ the idea in [19, Remark 5.1] to solve the HJB-QVI under the original $z = e^w$ grid using a finite difference method. For each time τ_m, numerical solutions for $w \in [w^\dagger_{\min}, w'_{\min}]$ (obtained by finite difference method) and for $w \in (w'_{\min}, w_{\max}]$ (obtained by our scheme) can be combined to compute τ_{m+1} solutions in $(w_{\min}, w_{\max})$. This approach allows for a consistency proof essentially the same. It is also noteworthy that we show good numerical results in Section 4 without imposing the condition (5.27).*

Remark 5.4. *It can be verified that, for a smooth test function $\phi(\mathbf{x})$, the operator $F_{in}(\mathbf{x}, p_1, p_2, p_3, p_4, p_5)$, defined in* (3.10)*, is continuous in its parameters, i.e. continuous in* $(\mathbf{x}, p_1, p_2, p_3, p_4, p_5)$*. The same continuity property also holds for operators $F_{a_{\min}}(\mathbf{x}, p_1, p_2, p_3, p_4)$, $F_{w_{\min}}(\mathbf{x}, p_1, p_2, p_5)$, $F_{wa_{\min}}(\mathbf{x}, p_1, p_2)$, $F_{w_{\max}}(\mathbf{x}, p_1)$, $F_{\tau_0}(\mathbf{x}, p_1)$, respectively defined in* (3.11)-(3.15)*.*

We now verify the consistency of scheme (5.25). We first define the notion of consistency in the viscosity sense below.

Definition 5.1 (Consistency in viscosity sense)**.** *Suppose the discretization parameter h satisfies* (4.10)*. The numerical scheme* (5.25) *is consistent in the viscosity sense if, for all $\hat{\mathbf{x}} = (\hat{w}, \hat{a}, \hat{\tau}) \in \Omega^\infty$, and for any $\phi \in \mathcal{G}(\Omega^\infty) \cap \mathcal{C}^\infty(\Omega^\infty)$, with $\phi_{n,j}^m = \phi\left(\mathbf{x}_{n,j}^m\right)$ and $\mathbf{x} = (w_n, a_j, \tau_{m+1})$, we have both of the following*

$$\limsup_{\substack{h\to 0,\ \mathbf{x}\to\hat{\mathbf{x}} \\ \xi\to 0}} \mathcal{H}_{n,j}^{m+1}\Big(h, \phi_{n,j}^{m+1}+\xi, \left\{\phi_{l,k}^m+\xi\right\}_{k\le j}\Big) \le (F_{\Omega^\infty})^*\big(\hat{\mathbf{x}}, \phi(\hat{\mathbf{x}}), D\phi(\hat{\mathbf{x}}), D^2\phi(\hat{\mathbf{x}}), \mathcal{J}\phi(\hat{\mathbf{x}}), \mathcal{M}\phi(\hat{\mathbf{x}})\big), \quad (5.52)$$

$$\liminf_{\substack{h\to 0,\ \mathbf{x}\to\hat{\mathbf{x}} \\ \xi\to 0}} \mathcal{H}_{n,j}^{m+1}\Big(h, \phi_{n,j}^{m+1}+\xi, \left\{\phi_{l,k}^m+\xi\right\}_{k\le j}\Big) \ge (F_{\Omega^\infty})_*\big(\hat{\mathbf{x}}, \phi(\hat{\mathbf{x}}), D\phi(\hat{\mathbf{x}}), D^2\phi(\hat{\mathbf{x}}), \mathcal{J}\phi(\hat{\mathbf{x}}), \mathcal{M}\phi(\hat{\mathbf{x}})\big). \quad (5.53)$$

Below, we state and prove the main lemma on consistency of scheme (5.25).

Lemma 5.5 (Consistency)**.** *Assuming all the conditions in Lemma 5.2 are satisfied, then the scheme* (5.25) *is consistent with the impulse control problem* (3.1) *in Ω^∞ in the sense of Definition 5.1.*

Proof of Lemma 5.5. We first prove (5.52). There exists sequences $\{h_i\}_i$, $\{n_i\}$, $\{j_i\}$, $\{m_i\}$, and $\{\xi_i\}$, such that

$$h_i \to 0,\ \xi_i \to 0,\ \mathbf{x}_i \equiv (w_{n_i}, a_{j_i}, \tau_{m_i+1}) \to \hat{\mathbf{x}} \equiv (\hat{w}, \hat{a}, \hat{\tau}), \quad \text{as} \quad i \to \infty, \quad (5.54)$$

and

$$\limsup_{i\to\infty} \mathcal{H}_{n_i,j_i}^{m_i+1}\left(h_i, \phi_{n_i,j_i}^{m_i+1}+\xi_i, \left\{\phi_{l_i,k_i}^{m_i}+\xi_i\right\}_{k_i\le j_i}\right) = \limsup_{\substack{h\to 0,\ \mathbf{x}\to\hat{\mathbf{x}} \\ \xi\to 0}} \mathcal{H}_{n,j}^{m+1}\left(h, \phi_{n,j}^{m+1}+\xi, \left\{\phi_{l,k}^m+\xi\right\}_{k\le j}\right). \quad (5.55)$$

We first consider the case $\hat{\mathbf{x}} \in \Omega_{\text{in}}$. Denote by $\Delta\tau_i$ the time step associated with the parameter h_i. For sufficiently small h_i, we have

$$w_{\min} < w_{n_i} < w_{\max}, \quad C_r\Delta\tau_i < a_{j_i} \le a_J, \quad \text{and} \quad 0 < \tau_{m_i+1} \le T.$$

According to the first equation of (5.28) in Lemma 5.2, we have

$$\begin{aligned} &\mathcal{H}_{n_i,j_i}^{m_i+1}\left(h_i, \phi_{n_i,j_i}^{m_i+1}+\xi_i, \left\{\phi_{l_i,k_i}^{m_i}+\xi_i\right\}_{k_i\le j_i}\right) \quad (5.56)\\ &\quad = F_{\text{in}}\left(\mathbf{x}_i, \phi(\mathbf{x}_i), D\phi(\mathbf{x}_i), D^2\phi(\mathbf{x}_i), \mathcal{J}\phi(\mathbf{x}_i), \mathcal{M}\phi(\mathbf{x}_i)\right) + c(\mathbf{x}_i)\,\xi_i + \mathcal{O}(h_i) + \mathcal{E}\left(\mathbf{x}_{n_i,j_i}^{m_i}, h_i\right). \end{aligned}$$

Combining (5.55) and (5.56), for $\hat{\mathbf{x}} \in \Omega_{\text{in}}$, with continuity of F_{in} (see Remark 5.4), we have

$$\begin{aligned} \limsup_{\substack{h\to 0,\ \mathbf{x}\to\hat{\mathbf{x}} \\ \xi\to 0}} \mathcal{H}_{n,j}^{m+1}\Big(h, \phi_{n,j}^{m+1}+\xi, \left\{\phi_{l,k}^m+\xi\right\}_{k\le j}\Big) &\le \limsup_{i\to\infty} F_{\text{in}}\left(\mathbf{x}_i, \phi(\mathbf{x}_i), D\phi(\mathbf{x}_i), D^2\phi(\mathbf{x}_i), \mathcal{J}\phi(\mathbf{x}_i), \mathcal{M}\phi(\mathbf{x}_i)\right)\\ &\quad + \limsup_{i\to\infty}\Big[c(\mathbf{x}_i)\,\xi_i + \mathcal{O}(h_i) + \mathcal{E}(\mathbf{x}_{n_i,j_i}^{m_i}, h_i)\Big]\\ &= F_{\text{in}}\left(\hat{\mathbf{x}}, \phi(\hat{\mathbf{x}}), D\phi(\hat{\mathbf{x}}), D^2\phi(\hat{\mathbf{x}}), \mathcal{J}\phi(\hat{\mathbf{x}}), \mathcal{M}\phi(\hat{\mathbf{x}})\right)\\ &= (F_{\Omega^\infty})^*\left(\hat{\mathbf{x}}, \phi(\hat{\mathbf{x}}), D\phi(\hat{\mathbf{x}}), D^2\phi(\hat{\mathbf{x}}), \mathcal{J}\phi(\hat{\mathbf{x}}), \mathcal{M}\phi(\hat{\mathbf{x}})\right). \end{aligned}$$

This proves (5.52) for $\hat{\mathbf{x}} \in \Omega_{\text{in}}$.

We define $\Omega_{bd} = \{w_{\min} \cup w_{\max}\} \times [a_{\min}, a_{\max}] \times (0, T]$. Following similar steps, (5.52) can be proved for $\hat{\mathbf{x}} \in \Omega_{w_{\min}}^\infty \setminus \Omega_{bd}$, $\hat{\mathbf{x}} \in \Omega_{w_{\max}}^\infty \setminus \Omega_{bd}$, and $\hat{\mathbf{x}} \in \Omega_{\tau_0}^\infty$, leaving $\hat{\mathbf{x}} \in \Omega_{bd}$ as a special case to be addressed below.

We now show (5.52) for special cases, namely $\hat{\mathbf{x}} \in \Omega_{a_{\min}}$, $\hat{\mathbf{x}} \in \Omega^{\infty}_{wa_{\min}}$, and $\hat{\mathbf{x}} \in \Omega_{bd}$. First, we consider $\hat{\mathbf{x}} \in \Omega_{a_{\min}}$. For the sequence $\{\mathbf{x}_i\} \to \hat{\mathbf{x}}$, we cannot guarantee $a_{j_i} \le C_r \Delta\tau_i$ or $a_{j_i} > C_r\Delta\tau_i$ even for a sufficiently small h_i. According to (5.28) in Lemma 5.2, $\mathcal{H}^{m_i+1}_{n_i,j_i}(\cdot)$ is given by

$$\mathcal{H}^{m_i+1}_{n_i,j_i}\left(h_i, \phi^{m_i+1}_{n_i,j_i} + \xi_i, \left\{\phi^{m_i}_{l_i,k_i} + \xi_i\right\}_{k_i \le j_i}\right) \tag{5.57}$$

$$= \begin{cases} F_{\text{in}}\left(\mathbf{x}_i, \phi(\mathbf{x}_i), D\phi(\mathbf{x}_i), D^2\phi(\mathbf{x}_i), \mathcal{J}\phi(\mathbf{x}_i), \mathcal{M}\phi(\mathbf{x}_i)\right) + c\left(\mathbf{x}_i\right)\xi_i + \mathcal{O}(h_i) + \ \mathcal{E}(\mathbf{x}^{m_i}_{n_i,j_i}, h_i), \\ \qquad \text{if } \ w_{\min} < w_{n_i} < w_{\max},\ C_r\Delta\tau_i < a_{j_i} \le a_J,\ 0 < \tau_{m_i+1} \le T \\ \\ F_{\text{in}'}\left(\mathbf{x}_i, \phi(\mathbf{x}_i), D\phi(\mathbf{x}_i), D^2\phi(\mathbf{x}_i), \mathcal{J}\phi(\mathbf{x}_i)\right) + c\left(\mathbf{x}_i\right)\xi_i + \ \mathcal{O}(h_i) + \ \mathcal{E}(\mathbf{x}^m_{n,j}, h), \\ \qquad \text{if } \ w_{\min} \ < \ w_{n_i} < w_{\max},\ 0 < a_{j_i} \le C_r\Delta\tau_i,\ 0 < \tau_{m_i+1} \le T \\ \\ F_{a_{\min}}\left(\mathbf{x}_i, \phi(\mathbf{x}_i), D\phi(\mathbf{x}_i), D^2\phi(\mathbf{x}_i), \mathcal{J}\phi(\mathbf{x}_i)\right) + c\left(\mathbf{x}_i\right)\xi_i + \mathcal{O}(h_i), \\ \qquad \text{if } \ w_{\min} \ < \ w_{n_i} < w_{\max},\ a_{j_i} = 0,\ 0 < \tau_{m_i+1} \le T. \end{cases}$$

Note that the right hand side of (5.57) contains $F_{\text{in}'}(\cdot)$, which is problematic since this operator is not part of F_{Ω^∞}. To handle this, we note that $\sup_{\hat{\gamma}\in[0,a/\Delta\tau]} \hat{\gamma}\left(1 - e^{-w}\phi_w - \phi_a\right) \ge 0$. Using this with the definition of $F_{a_{\min}}(\cdot)$ and $F_{\text{in}'}(\cdot)$ in (3.11) and (5.26), respectively, for $a_{\min} < a_{j_i} \le C_r\Delta\tau_i$, we obtain

$$F_{\text{in}'}\left(\mathbf{x}_i, \phi(\mathbf{x}_i), D\phi(\mathbf{x}_i), D^2\phi(\mathbf{x}_i), \mathcal{J}\phi(\mathbf{x}_i)\right) \le F_{a_{\min}}\left(\mathbf{x}_i, \phi(\mathbf{x}_i), D\phi(\mathbf{x}_i), D^2\phi(\mathbf{x}_i), \mathcal{J}\phi(\mathbf{x}_i)\right).$$

Using this result to eliminate $F_{\text{in}'}(\cdot)$ from $\limsup \mathcal{H}^{m+1}_{n,j}(\cdot)$ gives

$$\begin{aligned} \limsup_{\substack{h\to 0,\ \mathbf{x}\to\hat{\mathbf{x}} \\ \xi\to 0}} \mathcal{H}^{m+1}_{n,j}\left(h, \phi^{m+1}_{n,j} + \xi, \left\{\phi^m_{l,k} + \xi\right\}_{k\le j}\right) &\le \limsup_{i\to\infty} F_{\Omega^\infty}\left(\mathbf{x}_i, \phi(\mathbf{x}_i), D\phi(\mathbf{x}_i), D^2\phi(\mathbf{x}_i), \mathcal{J}\phi(\mathbf{x}_i), \mathcal{M}\phi(\mathbf{x}_i)\right) \\ &\quad + \limsup_{i\to\infty}\left[c\left(\mathbf{x}_i\right)\xi_i + \ \mathcal{E}(\mathbf{x}^{m_i}_{n_i,j_i}, h_i)\right] \\ &\le \left(F_{\Omega^\infty}\right)^*\left(\hat{\mathbf{x}}, \phi(\hat{\mathbf{x}}), D\phi(\hat{\mathbf{x}}), D^2\phi(\hat{\mathbf{x}}), \mathcal{J}\phi(\hat{\mathbf{x}}), \mathcal{M}\phi(\hat{\mathbf{x}})\right), \end{aligned}$$

which proves (5.52) for $\hat{\mathbf{x}} \in \Omega_{a_{\min}}$. Other special cases are treated similarly.

We now prove (5.53) for $\hat{\mathbf{x}} \in \Omega^\infty$, which can be proven in the same manner except the case $\hat{\mathbf{x}} \in \Omega_{a_{\min}}$, $\hat{\mathbf{x}} \in \Omega^\infty_{wa_{\min}}$. For brevity, we only show (5.53) for $\hat{\mathbf{x}} \in \Omega_{a_{\min}}$ here. The other special cases can be tackled similarly. There exists sequences $\{h_i\}$, $\{n_i\}$, $\{j_i\}$, $\{m_i\}$, and $\{\xi_i\}$ satisfying (5.54) and

$$\liminf_{i\to\infty} \mathcal{H}^{m_i+1}_{n_i,j_i}\left(h_i, \phi^{m_i+1}_{n_i,j_i} + \xi_i, \left\{\phi^{m_i}_{l_i,k_i} + \xi_i\right\}_{k_i\le j_i}\right) = \liminf_{\substack{h\to 0,\ \mathbf{x}\to\hat{\mathbf{x}} \\ \xi\to 0}} \mathcal{H}^{m+1}_{n,j}\left(h, \phi^{m+1}_{n,j} + \xi, \left\{\phi^m_{l,k} + \xi\right\}_{k\le j}\right). \tag{5.58}$$

Then, for sufficiently large i, (5.57) holds as discussed above. If $0 < a_{j_i} \le C_r\Delta\tau_i$, we observe

$$\sup_{\hat{\gamma}\in[0, a_{j_i}/\Delta\tau_i]} \hat{\gamma}\left(1 - e^{-w_{n_i}}\phi_w\left(\mathbf{x}_i\right) - \phi_a\left(\mathbf{x}_i\right)\right) \le \sup_{\hat{\gamma}\in[0,C_r]} \hat{\gamma}\left(1 - e^{-w_{n_i}}\phi_w\left(\mathbf{x}_i\right) - \phi_a\left(\mathbf{x}_i\right)\right),$$

which implies that

$$F_{\text{in}'}\left(\mathbf{x}_i, \phi(\mathbf{x}_i), D\phi(\mathbf{x}_i), D^2\phi(\mathbf{x}_i), \mathcal{J}\phi(\mathbf{x}_i)\right) \ge F_{\text{in}}\left(\mathbf{x}_i, \phi(\mathbf{x}_i), D\phi(\mathbf{x}_i), D^2\phi(\mathbf{x}_i), \mathcal{J}\phi(\mathbf{x}_i), \mathcal{M}\phi(\mathbf{x}_i)\right).$$

Using this result to eliminate $F_{\text{in}'}(\cdot)$ from $\liminf \mathcal{H}^{m+1}_{n,j}(\cdot)$ gives

$$\begin{aligned} \liminf_{\substack{h\to 0,\ \mathbf{x}\to\hat{\mathbf{x}} \\ \xi\to 0}} \mathcal{H}^{m+1}_{n,j}\left(h, \phi^{m+1}_{n,j} + \xi, \left\{\phi^m_{l,k} + \xi\right\}_{k\le l}\right) &\ge \liminf_{i\to\infty} F_{\Omega^\infty}\left(\mathbf{x}_i, \phi(\mathbf{x}_i), D\phi(\mathbf{x}_i), D^2\phi(\mathbf{x}_i), \mathcal{J}\phi(\mathbf{x}_i), \mathcal{M}\phi(\mathbf{x}_i)\right) \\ &\quad + \liminf_{i\to\infty}\left[c\left(\mathbf{x}_i\right)\xi_i + \ e\left(\mathbf{x}^{m_i}_{n_i,j_i}, h_i\right)\right] \\ &\ge \left(F_{\Omega^\infty}\right)_*\left(\hat{\mathbf{x}}, \phi(\hat{\mathbf{x}}), D\phi(\hat{\mathbf{x}}), D^2\phi(\hat{\mathbf{x}}), \mathcal{J}\phi(\hat{\mathbf{x}}), \mathcal{M}\phi(\hat{\mathbf{x}})\right). \end{aligned}$$

This concludes the proof. □

5.3 Monotonicity

We present a result on the monotonicity of scheme (5.25).

Lemma 5.6 (ϵ-monotonicity)**.** *If linear interpolation is used and the weight $\tilde{g}_{n-l}$ satisfies the monotonicity condition (4.33), i.e. $\Delta w \sum_{l=-N^\dagger/2}^{N^\dagger/2-1} \left| \min\left(\tilde{g}_{n-l}, 0\right) \right| \; < \; \epsilon \frac{\Delta \tau}{T}$, where $\epsilon > 0$, then scheme (5.25) satisfies*

$$\mathcal{H}_{n,j}^{m+1}\left(h, v_{n,j}^{m+1}, \{x_{l,k}^m\}_{k\le j}\right) \;\le\; \mathcal{H}_{n,j}^{m+1}\left(h, v_{n,j}^{m+1}, \{y_{l,k}^m\}_{k\le j}\right) \;+\; K\epsilon \tag{5.59}$$

for bounded $\{x_{l,k}^m\}$ and $\{y_{l,k}^m\}$ having $\{x_{l,k}^m\} \;\ge\; \{y_{l,k}^m\}$, where the inequality is understood in the component-wise sense, and K is a positive constant independent of h and ϵ.

Proof. It is straightforward to show $\mathcal{A}_{n,j}^{m+1}(\cdot)$ and $\mathcal{B}_{n,j}^{m+1}(\cdot)$, defined in (5.23), are strictly monotone, i.e.

$$\mathcal{A}_{n,j}^{m+1}(\cdot,\cdot,\{x_{l,k}^m\}_{k\le j}) \le \mathcal{A}_{n,j}^{m+1}(\cdot,\cdot,\{y_{l,k}^m\}_{k\le j}), \quad \mathcal{B}_{n,j}^{m+1}(\cdot,\cdot,\{x_{l,k}^m\}_{k\le j}) \le \mathcal{B}_{n,j}^{m+1}(\cdot,\cdot,\{y_{l,k}^m\}_{k\le j}). \tag{5.60}$$

The proof then boils down to proving ϵ-monotonicity for $\mathcal{C}_{n,j}^{m+1}(\cdot)$ and $\mathcal{D}_{n,j}^{m+1}(\cdot)$, defined in (5.24). Recall the linear interpolation operator $\mathcal{I}\{\cdot\}(\cdot)$ in (4.13)-(4.17). Let $\tilde{x}_{n,j}^m$ and $\tilde{y}_{n,j}^m$ be the results of the linear operators $\mathcal{I}\{x^m\}(\cdot)$ and $\mathcal{I}\{y^m\}(\cdot)$ acting on $\left\{\left(\left(w_l, a_k\right), x_{l,k}^m\right)\right\}$, and $\left\{\left(\left(w_l, a_k\right), y_{l,k}^m\right)\right\}$, respectively. We also define for $(x_{loc})_{n,j}^{m+}$, $(x_{nlc})_{n,j}^{m+}$, $(y_{loc})_{n,j}^{m+}$, and $(y_{nlc})_{n,j}^{m+}$ in a similar way that we define $(v_{loc})_{n,j}^{m+}$, $(v_{nlc})_{n,j}^{m+}$ in (4.18).

For the rest of the proof, let K be a generic positive constant independent of h and ϵ, which may take different values from line to line. From the boundedness of $\{x_{l,k}^m\}$ and $\{y_{l,k}^m\}$, and $\{x_{l,k}^m\} \;\ge\; \{y_{l,k}^m\}$, noting $\mathcal{I}\{x^m\}(\cdot)$ and $\mathcal{I}\{y^m\}(\cdot)$ are linear interpolation operators, we have, for all $l = -N^\dagger/2, \ldots, N^\dagger/2 - 1$,

$$(y_{loc})_{l,j}^{m+} \;\le\; (x_{loc})_{l,j}^{m+} \quad \text{and} \quad \left|(y_{loc})_{l,j}^{m+} - (x_{loc})_{l,j}^{m+}\right| \;\le\; K, \tag{5.61}$$

$$(y_{nlc})_{l,j}^{m+} \;\le\; (x_{nlc})_{l,j}^{m+} \quad \text{and} \quad \left|(y_{nlc})_{l,j}^{m+} - (x_{nlc})_{l,j}^{m+}\right| \;\le\; K, \tag{5.62}$$

where K is a positive constant independent of h and ϵ.

Next, using (5.61) together with the definition of the operator $\mathcal{C}_{n,j}^{m+1}(\cdot)$ in (5.24), we have

$$\begin{aligned}
&\mathcal{C}_{n,j}^{m+1}\left(h, v_{n,j}^{m+1}, \{x_{l,k}^m\}_{k\le j}\right) - \mathcal{C}_{n,j}^{m+1}\left(h, v_{n,j}^{m+1}, \{y_{l,k}^m\}_{k\le j}\right)\\
&= \frac{1}{\Delta\tau}\left[v_{n,j}^{m+1} - \Delta w \sum_{l=-N^\dagger/2}^{N^\dagger/2-1} \tilde{g}_{n-l}\;(x_{loc})_{l,j}^{m+}\right] - \frac{1}{\Delta\tau}\left[v_{n,j}^{m+1} - \Delta w \sum_{l=-N^\dagger/2}^{N^\dagger/2-1} \tilde{g}_{n-l}\;(y_{loc})_{l,j}^{m+}\right]\\
&\le \frac{1}{\Delta\tau}\left[\Delta w \sum_{l=-N^\dagger/2}^{N^\dagger/2-1} \left|\min\left(\tilde{g}_{n-l},0\right)\right| \left|(y_{loc})_{l,j}^{m+} - (x_{loc})_{l,j}^{m+}\right|\right]\\
&\le \frac{K}{\Delta\tau}\left(\Delta w \sum_{l=-N^\dagger/2}^{N^\dagger/2-1} \left|\min\left(\tilde{g}_{n-l},0\right)\right|\right) \;\le\; \epsilon\frac{K}{T},
\end{aligned} \tag{5.63}$$

where the last equality uses (4.33).

Similarly, using (5.62) together with the definition of the operator $\mathcal{D}_{n,j}^{m+1}(\cdot)$ in (5.24) yields

$$\begin{aligned}
&\mathcal{D}_{n,j}^{m+1}\left(h, v_{n,j}^{m+1}, \{x_{l,k}^m\}_{k\le j}\right) - \mathcal{D}_{n,j}^{m+1}\left(h, v_{n,j}^{m+1}, \{y_{l,k}^m\}_{k\le j}\right)\\
&\qquad \le \Delta w \sum_{l=-N^\dagger/2}^{N^\dagger/2-1} \left|\min\left(\tilde{g}_{n-l},0\right)\right| \left|(y_{nlc})_{l,j}^{m+} - (x_{nlc})_{l,j}^{m+}\right| \;\le\; \epsilon\frac{K\Delta\tau}{T}.
\end{aligned} \tag{5.64}$$

Putting (5.60), (5.63) and (5.64) together concludes the proof. □

5.4 Convergence to viscosity solution

We have demonstrated that the scheme (5.25) satisfies the three key properties in Ω: (i) ℓ_∞-stability (Lemma 5.1), (ii) consistency (Lemma 5.5) and (iii) ϵ-monotonicity (Lemma 5.6). With a provable strong comparison principle result for $\Omega_{\text{in}} \cup \Omega_{a_{\min}}$, we now present the main convergence result of the paper.

Theorem 5.1 (Convergence in $\Omega_{\text{in}} \cup \Omega_{a_{\min}}$)**.** *Suppose that all the conditions for Lemmas 5.1, 5.5 and 5.6 are satisfied. Under the assumption that the monotonicity tolerance $\epsilon \to 0$ as $h \to 0$, scheme* (5.25) *converges locally uniformly in $\Omega_{in} \cup \Omega_{a_{\min}}$ to the unique bounded viscosity solution of the GMWB pricing problem in the sense of Definition 3.2.*

Proof. To clearly indicate the important role of the discretization parameter h, in this proof, we use $\mathbf{x}_{n,j}^{m+1}(h) = (w_n, a_j, \tau_{m+1}; h)$. Furthermore, we use $v_{n,j}^{m+1}(h)$ to denote the numerical solution at the node $\mathbf{x}_{n,j}^{m+1}(h)$. We define the u.s.c. (respectively l.s.c.) function $\overline{v} : \Omega^\infty \to \mathbb{R}$ (respectively $\underline{v} : \Omega^\infty \to \mathbb{R}$) by

$$\overline{v}\,(\mathbf{x}) = \limsup_{\substack{h\to 0\\ \mathbf{x}_{n,j}^{m+1}(h)\to\mathbf{x}}} v_{n,j}^{m+1}(h) \qquad (\text{resp. } \underline{v}(\mathbf{x}) = \liminf_{\substack{h\to 0\\ \mathbf{x}_{n,j}^{m+1}(h)\to\mathbf{x}}} v_{n,j}^{m+1}(h)) \qquad \mathbf{x} \in \Omega^\infty. \tag{5.65}$$

We now show that $\overline{v}(\mathbf{x})$ (resp. $\underline{v}(\mathbf{x})$) is a subsolution (resp. supersolution) in Ω^∞ in the sense of Definition 3.2. By stability of our scheme in Ω^∞ established in Lemma 5.1, functions $\overline{v}$ and $\underline{v}$ are in $\mathcal{G}(\Omega^\infty)$. Since definition (5.65) implies that $\overline{v}^*(\mathbf{x}) = \overline{v}(\mathbf{x})$ and $\underline{v}_*(\mathbf{x}) = \underline{v}(\mathbf{x})$ for all $\mathbf{x} \in \Omega^\infty$, we will work with $\overline{v}(\mathbf{x})$ and $\underline{v}(\mathbf{x})$ instead of their respective envelopes.

For the case $\overline{v}(\mathbf{x})$, we let $\hat{\mathbf{x}} \in \Omega^\infty$ be fixed, and $\phi \in \mathcal{G}(\Omega^\infty) \cap \mathcal{C}^\infty(\Omega^\infty)$ such that (i) $(\overline{v} - \phi)(\mathbf{x})$ has a global maximum on Ω^∞ at $\mathbf{x} = \hat{\mathbf{x}}$, and (ii) $\phi\,(\hat{\mathbf{x}}) = \overline{v}\,(\hat{\mathbf{x}})$. That is, $\phi\,(\mathbf{x})$ satisfies

$$\begin{cases} \phi\,(\mathbf{x}) & > & \overline{v}\,(\mathbf{x})\,, \quad \forall\, \mathbf{x} \in \Omega^\infty \text{ and } \mathbf{x} \neq \hat{\mathbf{x}}, \\ \phi\,(\mathbf{x}) & = & \overline{v}\,(\mathbf{x})\,, \quad \mathbf{x} = \hat{\mathbf{x}}. \end{cases} \tag{5.66}$$

Consider a sequence of grids with discretization parameter h_i such that $h_i \to 0$ as $i \to \infty$. We denote by Ω_{h_i} the grid parameterized by h_i, noting that $\Omega_{h_i} \to \Omega^\infty$ as $i \to \infty$. Let $\mathbf{x}_{n_i,j_i}^{m_i+1}(h_i) \equiv (w_{n_i}, a_{j_i}, \tau_{m_i+1}; h_i)$ be a node in Ω^∞ such that

$$v_{n_i,j_i}^{m_i+1}\,(h_i) - \phi_{n_i,j_i}^{m_i+1}\,(h_i) \;\text{ is a global maximum on }\; \Omega_{h_i}, \tag{5.67}$$

where $\phi\,(\mathbf{x})$ is the test function satisfying (5.66), with the usual notation $\phi_{n_i,j_i}^{m_i+1}\,(h_i) = \phi\left(\mathbf{x}_{n_i,j_i}^{m_i+1}\,(h_i)\right)$. First, we note that

$$\mathbf{x}_{n_i,j_i}^{m_i+1}(h_i) \to \hat{\mathbf{x}} \quad \text{and also} \quad \mathbf{x}_{n_i,j_i}^{m_i}(h_i) \to \hat{\mathbf{x}}, \quad \text{as} \quad i \to \infty. \tag{5.68}$$

In addition, for any finite discretization parameter h_i, the global maximum in (5.67) is not necessarily zero, as $\mathbf{x}_{n_i,j_i}^{m_i+1}(h_i) = \hat{\mathbf{x}}$ is not necessarily true. Since $\phi(\cdot)$ satisfies (5.66), we have

$$v_{n_i,j_i}^{m_i+1}\,(h_i) = \phi_{n_i,j_i}^{m_i+1}\,(h_i) + \xi_i, \quad \text{where} \quad \xi_i \to 0, \quad \text{as} \quad i \to \infty. \tag{5.69}$$

Because the global maximum (5.67) is attained at $\mathbf{x}_{n_i,j_i}^{m_i+1}(h_i)$, we have that, for all l_i and k_i used in the scheme $\mathcal{H}_{n_i,j_i}^{m_i+1}\left(h_i, v_{n_i,j_i}^{m_i+1}(h_i), \left\{v_{l_i,k_i}^{m_i}(h_i)\right\}_{k_i \le j_i}\right)$, we have

$$v_{l_i,k_i}^{m_i}\,(h_i) - \phi_{l_i,k_i}^{m_i}\,(h_i) \;\le\; v_{n_i,j_i}^{m_i+1}\,(h_i) - \phi_{n_i,j_i}^{m_i+1}\,(h_i) \;=\; \xi_i, \tag{5.70}$$

where ξ_i is defined in (5.69). Using (5.69), (5.70), and the monotonicity result in Lemma 5.6, we obtain

$$\begin{aligned} 0 \;=\; & \mathcal{H}_{n_i,j_i}^{m_i+1}\left(h_i, v_{n_i,j_i}^{m_i+1}(h_i), \left\{v_{l_i,k_i}^{m_i}(h_i)\right\}_{k_i \le j_i}\right) \\ & \ge\; \mathcal{H}_{n_i,j_i}^{m_i+1}\left(h_i, \phi_{n_i,j_i}^{m_i+1}\,(h_i) + \xi_i, \left\{\phi_{l_i,k_i}^{m_i}\,(h_i) + \xi_i\right\}_{k_i \le j_i}\right) - C\epsilon_i, \end{aligned} \tag{5.71}$$

where $C > 0$ and $\epsilon_i \to 0$, as $i \to \infty$.

Letting $i \to \infty$ and using the consistency result from Lemma 5.5, (5.71) gives

$$0 \geq \liminf_{i\to\infty} \mathcal{H}^{m_i+1}_{n_i,j_i}\left(h_i, \phi^{m_i+1}_{n_i,j_i}(h_i) + \xi_i, \left\{\phi^{m_i}_{l_i,k_i}(h_i) + \xi_i\right\}_{k_i \leq j_i}\right) - \liminf_{i\to\infty} C\epsilon_i$$
$$\geq (F_{\Omega^\infty})_* \left(\hat{\mathbf{x}}, \phi(\hat{\mathbf{x}}), D\phi(\hat{\mathbf{x}}), D^2\phi(\hat{\mathbf{x}}), \mathcal{J}\phi(\hat{\mathbf{x}}), \mathcal{M}\phi(\hat{\mathbf{x}})\right).$$

This shows that $\overline{v}(\mathbf{x})$ is a subsolution in Ω^∞ in the sense of Definition 3.2. A similar argument shows that $\underline{v}(\mathbf{x})$ is a supersolution in Ω^∞. By definition of $\overline{v}(\mathbf{x})$ and $\underline{v}(\mathbf{x})$ in (5.65), we have that $\overline{v}(\mathbf{x}) \geq \underline{v}(\mathbf{x})$, $\forall \mathbf{x} \in \Omega^\infty$. Since a strong comparison principle result holds in $\Omega_{\text{in}} \cup \Omega_{a_{\min}}$, we have $\overline{v}(\mathbf{x}) \leq \underline{v}(\mathbf{x})$, $\forall \mathbf{x} \in \Omega_{\text{in}} \cup \Omega_{a_{\min}}$. Therefore, $v(\mathbf{x}) = \overline{v}(\mathbf{x}) = \underline{v}(\mathbf{x})$ is the unique viscosity solution in $\Omega_{\text{in}} \cup \Omega_{a_{\min}}$. As a result,

$$v(\mathbf{x}) = \lim_{\substack{h\to 0 \\ \mathbf{x}^{m+1}_{n,j}(h)\to\mathbf{x}}} v^{m+1}_{n,j}(h), \quad \text{for } \mathbf{x} \in \Omega_{\text{in}} \cup \Omega_{a_{\min}},$$

from which we obtain that convergence is locally uniform. □

6 Numerical examples

In this section, we provide selected numerical results of our ϵ-monotone Fourier method applied to the the impulse control GMWB pricing problem. For all experiments, unless otherwise noted, the details of the mesh size/timestep refinement levels used are given in Table 6.2. As noted previously, for practical purposes, if $P^\dagger$ is chosen sufficiently large, it can be kept constant for all refinement levels (as we let $h \to 0$). For our numerical experiments, we use $w_{\min} = \ln(z_0) - 10$ and $w_{\max} = \ln(z_0) + 10$, and $w^\dagger_{\min}$ and $w^\dagger_{\max}$ constructed as discussed in Remark 4.1, so $w_{\min} = \ln(z_0) - 20$ and $w^\dagger_{\max} = \ln(z_0) + 20$. Tests with larger intervals also show negligible effect on numerical solutions.

Our numerical prices are verified against those produced by two other methods, namely (i) Finite Difference (FD) methods ([19] and [40]), and (ii) Monte Carlo (MC) simulation. To carry out Monte Carlo validation, we proceed in two steps. In Step 1, we solve the GMWB pricing problem using the proposed ϵ-monotone Fourier method on a relatively fine computational grid (2^{12} w-nodes, 401 a-nodes, and 480 timesteps). During this step, the optimal controls are stored for each discrete state value and timestep. In Step 2, we carry out Monte Carlo simulations from $t = 0$ to $t = T$ following these stored PDE-computed optimal strategies, using linear interpolation, if necessary, to determine the controls for a given state value. For Step 2, a total of 10^6 paths is used.

Motivated by findings in [19], [40], a sufficiently small fixed cost $c = 10^{-8}$ is used all numerical tests. For user-defined tolerances ϵ and ϵ_1 in Algorithm (4.1), we use $\epsilon = \epsilon_1 = 10^{-6}$ for all refinement levels. Through numerical experiments, it is observed that using smaller ϵ or ϵ_1 produced virtually identical numerical results, indicating that this value of ϵ and ϵ_1 are sufficient for all practical purposes.

Parameter	Value
Expiry time (T)	10.0 years
Interest rate (r)	0.05
Maximum withdrawal rate (G_r)	10/year
Withdrawal penalty (μ)	0.10
Initial Lump-sum premium (z_0)	100
Initial guarantee account balance ($= z_0$)	100
Initial sub-account value ($= z_0$)	100

Table 6.1: *Common GMWB parameters used in the numerical tests*

Level	N (w)	J (a)	M (τ)
0	2^{10}	51	60
1	2^{11}	101	120
2	2^{12}	201	240
3	2^{13}	401	480
4	2^{14}	801	960

Table 6.2: *Grid and timestep refinement levels for numerical tests; $w_{\min} = \ln(z_0) - 10$ and $w_{\max} = \ln(z_0) + 10$; $w^\dagger_{\min}$ and $w^\dagger_{\max}$ constructed using* (4.7).

6.1 Validation examples

6.1.1 No Jumps – the GBM model

In this example, we repeat some numerical examples in [19] where (2.2) is a GBM. Table 6.3 presents convergence results for $\sigma = \{0.2, 0.3\}$, assuming a zero insurance fee and continuous withdrawal. To provide an estimate of the convergence rate of the algorithm, we compute the "Change" as the difference in values from the coarser grid and the "Ratio" as the ratio of changes between successive grids. The numerical results indicate that first-order convergence is achieved for the algorithm. Results obtained by MC simulation also indicate excellent agreement with those obtained by the proposed ϵ-monotone Fourier method

Method	Level	$\sigma = 0.20$			$\sigma = 0.30$		
		Value	Change	Ratio	Value	Change	Ratio
ϵ-monotone Fourier	0	107.7726			115.7736		
	1	107.7573	-0.0153		115.8422	0.0686	
	2	107.7481	-0.0092	1.65	115.8716	0.0294	2.33
	3	107.7423	-0.0058	1.59	115.8834	0.0118	2.49
	4	107.7391	-0.0032	1.83	115.8881	0.0047	2.50
FD		107.7313			115.8842		
MC	95%-CI	[107.6020, 107.8430]			[115.6192, 116.0480]		

TABLE 6.3: *Convergence study for the value of the GMWB guarantee at* $t = 0$, $z = a = 100$. *No insurance fee* ($\beta = 0$) *is imposed; FD benchmark value is from [19] (Table 3, finest grid).*

6.1.2 Jumps – log-normal

In this test, $\ln \psi$ is normally distributed with its density function $b(y)$ given by (2.3). Table 6.4 shows the parameters of the log-normal jump process, taken from [42]. Table 6.5 presents the convergence results with $\sigma = 0.3$, assuming a fair/no-arbitrage insurance fee of $\beta = 0.045452043$ and continuous withdrawal. As stated in [42], since the no-arbitrage fee is imposed, the exact price is 100. It is observed from Table 6.5 that numerical prices produced by our method exhibit (first-order) convergence to this exact price. Results obtained by MC simulation also indicate excellent agreement with those obtained by the proposed ϵ-monotone Fourier method.

Parameter	Value
ς	0.45
ν	-0.9
λ	0.1

TABLE 6.4: *Jump parameters for log-normal distribution*

Method	Level	Value	Change	Ratio
ϵ-monotone Fourier	0	100.2822		
	1	100.1391	-0.1432	
	2	100.0694	-0.0696	2.06
	3	100.0350	-0.0345	2.02
	4	100.0177	-0.0173	1.99
FD		100.00003		
MC	95%-CI	[99.9056, 100.1010]		

TABLE 6.5: *Convergence study for the value of the GMWB guarantee at* $t = 0$, $z = a = 100$. $\sigma = 0.3$ *and fair insurance fee* ($\beta = 0.045452043$) *is imposed; FD benchmark value is from [42] (Table 7.4, finest grid).*

6.1.3 Jumps – log-double-exponential

In this test, $\ln \psi$ is double-exponential distributed with its density function $b(y)$ given by (2.4). Table 6.6 shows the jump diffusion parameters. Since a reference price for this case is not available in the literature, we implement the FD scheme proposed in [19], originally developed for diffusion processes. For the finest grid (i.e. the level 5 grid and timestep data used in [19, Table 2]), the FD benchmark value in this case is 118.4130. Table 6.7 presents the convergence results $\sigma = 0.3$, assuming a zero insurance fee and continuous withdrawal. Results obtained by Monte Carlo simulation also indicate excellent agreement with those obtained by the FD and the proposed ϵ-monotone Fourier method

Parameter	Value
p_u	0.3445
η_1	3.0465
η_2	3.0775
λ	0.1

Table 6.6: *Jump parameters for log-double-exponential distribution*

Method	Level	Value	Change	Ratio
ϵ-monotone Fourier	0	118.3453		
	1	118.3905	0.0452	
	2	118.4097	0.0192	2.35
	3	118.4172	0.0075	2.56
	4	118.4200	0.0028	2.63
FD		118.4130		
MC	95%-CI	[118.1679, 118.7308]		

Table 6.7: *Convergence study for the value of the GMWB guarantee at $t = 0$, $z = a = 100$; $\sigma = 0.3$ and no insurance fee ($\beta = 0$).*

6.2 Wrap-around errors

6.2.1 Application of Theorem 4.1

In this experiment, we numerically illustrate that the proposed treatment of the wrap-around error is sufficient, i.e. the wrap-around error is bounded Theorem 4.1. For brevity, we present only results of the GBM case with $\sigma = 0.2$. Results of other cases are similar, and hence omitted.

First, we note that the condition (4.38) of Theorem 4.1 is satisfied due to stability by Lemma 5.1. To numerically check condition (4.39), using similar notations in Subsection 4.4, we denote

$$\text{SUM}_{\text{LEFT}} = \Delta w \sum_{\ell=-N^{\dagger}/2}^{-N/2-1} |\tilde{g}(\ell)|, \quad \text{SUM}_{\text{RIGHT}} = \Delta w \sum_{\ell=N/2+1}^{N^{\dagger}/2-1} |\tilde{g}(\ell)|, \quad \text{SUM} = \Delta w \sum_{\ell\in\mathbb{N}^{\dagger}} \tilde{g}(\ell).$$

Table 6.8 presents select results. Using the padding technique presented in Subsection 4.4, it is clear from Table 6.8 that the approximations of the Green's function on the left and right padding areas, namely the quantities SUM_{LEFT} and $\text{SUM}_{\text{RIGHT}}$, are negligible. It is worth noting that condition (4.39) is fulfilled for all refinement levels with the same user-specified numerical tolerance ϵ_e. Also from Table 6.8, it is clear that the total sum of the approximations of the Green's function approximately equals $e^{-r\Delta\tau}$ for each level, which agrees with (5.1).

Level	$\epsilon_e\Delta\tau/2$	SUM_{LEFT}	$\text{SUM}_{\text{RIGHT}}$	SUM
0	8.33333e-10	7.14037e-16	6.74673e-16	0.991701
1	4.16667e-10	8.71373e-16	7.75466e-16	0.995842
2	2.08333e-10	9.34340e-16	1.00408e-15	0.997919
3	1.04167e-10	1.17304e-15	1.15816e-15	0.998959
4	5.20833e-11	1.23246e-15	1.34286e-15	0.999479

Table 6.8: *The approximation of the Green's functions for the GBM model with $\epsilon_e = 10^{-8}$.*

6.2.2 Padding areas

Numerical results presented so far are based padding areas constructed via (4.7). In this experiment, we numerically demonstrate that larger padding areas are not needed. To this end, we use

$$w^{\dagger}_{\min} = w_{\min} - 1.5\left(w_{\max} - w_{\min}\right) \quad \text{and} \quad w^{\dagger}_{\max} = w_{\max} + 1.5\left(w_{\max} - w_{\min}\right),$$

and $N^{\dagger} = 4N$. For fair comparison, we utilize the same padding techniques and the same Δw with previous numerical tests, where (4.7) and $N^{\dagger} = 2N$ are employed. The numerical prices of this test are reported in Table 6.9 (col. "Value"). They are to be compared with numerical prices from Tables 6.3, 6.5, 6.7 (col. "Value"), which, for convenience, are also included in Table 6.9. It it evident from Table 6.9 that using a larger padding area virtually does not affect the numerical prices. This confirms that our choice of the padding areas in (4.7) is sufficiently suitable for practical purposes.

Level	GBM model				log-normal distribution		log-double-exp distribution	
	$\sigma = 0.20$		$\sigma = 0.30$					
	Value	Value (Tab. 6.3)	Value	Value (Tab. 6.3)	Value	Value (Tab. 6.5)	Value	Value (Tab. 6.7)
0	107.7726	107.7726	115.7735	115.7736	100.2823	100.2822	118.3451	118.3453
1	107.7574	107.7574	115.8420	115.8422	100.1390	100.1391	118.3903	118.3905
2	107.7481	107.7481	115.8714	115.8716	100.0696	100.0694	118.4096	118.4097
3	107.7423	107.7423	115.8832	115.8834	100.0352	100.0350	118.4172	118.4172
4	107.7391	107.7391	115.8879	115.8881	100.0180	100.0177	118.4201	118.4200

TABLE 6.9: *Prices obtained using larger padding areas with $\theta = 3$ in (4.7) and $N^{\dagger} = 4N$. Compare with prices in Table 6.3, 6.5, 6.7 where (4.7) is used and $N^{\dagger} = 2N$.*

6.2.3 Zero padding technique

We redo all the above experiments using the zero padding techniques proposed in [1, 45], and prices obtained from these experiments are presented in Table 6.10. These prices are to be compared with numerical prices from Tables 6.3, 6.5, 6.7 (col. "Value"), which, for convenience, are also included in Table 6.10.

Level	GBM model				log-normal distribution		log-double-exp distribution	
	$\sigma = 0.20$		$\sigma = 0.30$					
	Value	Value (Tab. 6.3)	Value	Value (Tab. 6.3)	Value	Value (Tab. 6.5)	Value	Value (Tab. 6.7)
0	107.4793	107.7726	115.3974	115.7736	99.7237	100.2822	117.9545	118.3453
1	107.4458	107.7574	115.4431	115.8422	99.5491	100.1391	117.9760	118.3905
2	107.4274	107.7481	115.4608	115.8716	99.4636	100.0694	117.9831	118.4097
3	107.4170	107.7423	115.4668	115.8834	99.4211	100.0350	117.9847	118.4172
4	107.4115	107.7391	115.4686	115.8881	99.3999	100.0177	117.9846	118.4200

TABLE 6.10: *Results using zero padding technique. Compare with results in Table 6.3, 6.5, 6.7 where the asymptotic boundary conditions are used.*

It is evident from Table 6.10 that numerical prices obtained using the zero padding technique do not converge to the same prices as those obtained using our padding techniques. Specifically, numerical prices in the former case are consistently smaller than our numerical prices, with the contamination appears to be more severe with jumps-diffusion models. This is expected as the zero padding technique tends to underprice a GMWB as $e^{w} \to 0$. These results indicate that the zero padding technique is not suitable for use in pricing GMWB.

7 Conclusion

In this paper, we develop an ϵ-monotone numerical Fourier method for the HJB-QVI associated with an impulse control formulation arising in the pricing of GMWB under jump-diffusion dynamics. We propose an efficient implementation of the scheme via FFT, including a proper handling of boundary conditions and padding techniques. We mathematically prove that our padding techniques can effectively control wraparound errors in the numerical solutions. We appeal to a Barles-Souganidis-type analysis in [14], to rigorously prove the convergence of our scheme the unique viscosity solution of the HJB-QVI as the discretization parameter and the monotonicity tolerance ϵ approach zero. Although we focus specifically on GMWB, our comprehensive and systematic approach could serve as a numerical and convergence analysis framework for the development of similar weakly monotone methods for HJB-QVIs arising from impulse control problems in finance.

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Appendix A Wraparound error

To avoid subscript clutter, in this appendix, we use the notation $\tilde{g}(n-l) \equiv \tilde{g}_{n-l}$ and $u^m(n) \equiv u^m_n$. Noting this notation, the equation (4.37) becomes the following generic recursion

$$u^m(n) \;=\; \Delta w \sum_{l=-N^\dagger/2}^{N^\dagger/2-1} \tilde{g}(n-l)\; u^{m-1}(l), \quad N^\dagger \in \{N, 2N, 4N, \ldots\}\,,$$

As an example of wraparound error, we examine a worst case term in equation (A.1) below. Consider the term in (A.1) corresponding to $n = -N/2+1$, which corresponds to the node having w adjacent to $w_{\min}$, and $l = N^\dagger/2-1$, namely

$$\Delta w \;\tilde{g}(-N/2+1-N^\dagger/2+1)\; u^{m-1}(N^\dagger/2-1). \tag{A.1}$$

By periodic extension, we shift the argument of $\tilde{g}(\cdot)$ by $N^\dagger$, resulting in

$$\tilde{g}(-N/2+1-N^\dagger/2+1) = \tilde{g}(-N/2+1-N^\dagger/2+1+N^\dagger) = \tilde{g}(-N/2+N^\dagger/2+2),$$

and hence, the term (A.1) becomes

$$\Delta w \;\tilde{g}(-N/2+N^\dagger/2+2)\; u^{m-1}(N^\dagger/2-1).$$

Hence, in this extreme case, equation (A.1) becomes

$$u^m(-N/2+1) \;=\; \Delta w \;\tilde{g}(-N/2+N^\dagger/2+2)\; u^{m-1}(N^\dagger/2-1) \;+\; \sum_{l=-N^\dagger/2}^{N^\dagger/2-2} (\text{ remaining terms }). \tag{A.2}$$

Example 1 (No padding: $N^\dagger = N$)**.** *Suppose we do not use any padding, so that that $N^\dagger = N$. In this case, equation* (A.2) *becomes*

$$u^m(-N/2+1) \;=\; \Delta w\ \tilde{g}(2)\ u^{m-1}(N/2-1)\ +\sum_{l=-N/2}^{N/2-2} (\ \textit{remaining terms}\ ). \tag{A.3}$$

Since, in general, $\tilde{g}(2)$ is not small, we can see that the term $u^{m-1}(N/2-1)$ has a considerable effect on $u^m(-N/2+1)$, which should not be the case. We can see here that the periodic extension of $\tilde{g}$ causes a wraparound *effect.*

Example 2 (Padding: $N^\dagger = 2N$)**.** *If $N^\dagger = 2N$, then equation* (A.2) *becomes*

$$u^m(-N/2+1) \;=\; \Delta w\ \tilde{g}(N/2+2)\ u^{m-1}(N^\dagger/2-1)\ +\sum_{l=-N^\dagger/2}^{N^\dagger/2-2} (\ \textit{other terms}\ ). \tag{A.4}$$

In this case, from (4.6)*, we have selected N sufficiently large so that $\tilde{g}(l) \simeq 0$, $l > N/2$ and $l < -N/2$, hence the leading term in equation* (A.4) *is small, and hence, wraparound error is reduced.*

Now we proceed to proving Theorem 4.1.

Proof. Using $|u_l^m| \le C$, $l \;=\; -N^\dagger/2, \ldots, N^\dagger/2-1$ and equation (4.38) gives

$$e_{\text{wrap}}^m \;\le\; C \max_n \left\{ \Delta w \sum_{l=-N^\dagger/2}^{N^\dagger/2-1} |\tilde{g}(n-l)| \left(\mathbf{1}_{\{(n-l)<-N^\dagger/2\}} + \mathbf{1}_{\{(n-l)>N^\dagger/2-1\}} \right) \right\}. \tag{A.5}$$

Recall that $n \in \{-N/2+1, \ldots, N/2-1\}$, hence the worst case values of n on the right hand side of equation (A.5) are $n = -N/2+1$ and $n = N/2-1$. Therefore, equation (A.5) gives

$$\begin{aligned} e_{\text{wrap}}^m \;&\le\; C\Delta w \sum_{l=-N^\dagger/2}^{N^\dagger/2-1} |\tilde{g}(N/2-1-l)| \mathbf{1}_{\{(N/2-1-l)>N^\dagger/2-1\}} \\ &\quad + C\Delta w \sum_{l=-N^\dagger/2}^{N^\dagger/2-1} |\tilde{g}(-N/2+1-l)|\ \mathbf{1}_{\{(-N/2+1-l)<-N^\dagger/2\}}. \end{aligned} \tag{A.6}$$

Also, since $N = N^\dagger/2$ equation (A.6) becomes

$$\begin{aligned} e_{\text{wrap}}^m \;&\le\; C\Delta w \sum_{l=-N^\dagger/2}^{N^\dagger/2-1} |\tilde{g}(N^\dagger/4-1-l)|\ \mathbf{1}_{\{(N^\dagger/4-1-l)>N^\dagger/2-1\}} \\ &\quad + C\Delta w \sum_{l=-N^\dagger/2}^{N^\dagger/2-1} |\tilde{g}(-N^\dagger/4+1-l)|\ \mathbf{1}_{\{(-N^\dagger/4+1-l)<-N^\dagger/2\}}, \end{aligned}$$

and eliminating the indicator functions gives

$$e_{\text{wrap}}^m \;\le\; C\Delta w \sum_{l=-N^\dagger/2}^{-N^\dagger/4-1} |\tilde{g}(N^\dagger/4-1-l)|\ + C\Delta w \sum_{l=N^\dagger/4+2}^{N^\dagger/2-1} |\tilde{g}(-N^\dagger/4+1-l)|.$$

Shifting $\tilde{g}(\cdot)$ by $\pm N^\dagger$ so that the argument of $\tilde{g}(\cdot)$ is in the range $[-N^\dagger/2, N^\dagger/2-1]$, implies

$$\begin{aligned} e_{\text{wrap}}^m \;&\le\; C\Delta w \sum_{l=-N^\dagger/2}^{-N^\dagger/4-1} |\tilde{g}(N^\dagger/4-1-l-N^\dagger)|\ + C\Delta w \sum_{l=N^\dagger/4+2}^{N^\dagger/2-1} |\tilde{g}(-N^\dagger/4+1-l+N^\dagger)| \\ &=\; C\Delta w \sum_{l=-N^\dagger/2}^{-N^\dagger/4-1} |\tilde{g}(-3N^\dagger/4-1-l)|\ + C\Delta w \sum_{l=N^\dagger/4+2}^{N^\dagger/2-1} |\tilde{g}(3N^\dagger/4+1-l)|. \end{aligned}$$

Rearranging the indices, gives

$$e_{\text{wrap}}^m \;\le\; C\Delta w \sum_{l=-N^\dagger/2}^{-N^\dagger/4-1} |\tilde{g}(l)|\ + C\Delta w \sum_{l=N^\dagger/4+2}^{N^\dagger/2-1} |\tilde{g}(l)|, \tag{A.7}$$

which, since $N = N^\dagger/2$, implies that equation (A.7) satisfies

$$
\begin{aligned}
e^m_{\text{wrap}} &\leq C\Delta w \sum_{l=-N^\dagger/2}^{-N/2-1} |\tilde{g}(l)| + C\Delta w \sum_{l=N/2}^{N^\dagger/2-1} |\tilde{g}(l)| \\
&= C\epsilon_e \Delta\tau, \qquad \text{(A.8)}
\end{aligned}
$$

where the last step follows from (4.39). Applying equation (A.8) recursively gives the bound $TC\epsilon_e$.

□

Appendix B Proof of a strong comparison principle

In this section, we prove a comparison principle in $\Omega_{\text{in}} \cup \Omega_{a_{\min}}$ for the GMWB impulse control pricing problem given in Definition 3.1. As the first step, in the next subsection, we will establish equivalence between relevant definitions of viscosity solutions for this problem.

B.1 Definitions of viscosity solution

For HJB-QVIs of the form (3.16), there are two alternative definitions of viscosity solution available in the literature. The first definition, previously presented in Definition 3.2 and reproduced in Definition B.1 below, is similar to [27, Definition 4.1], [6, Definition 2]. It appears that, for convergence analysis of a numerical scheme, it is often more convenient to use this definition.

Definition B.1 (Viscosity solution of equation (3.16))**.** *A locally bounded function $v \in \mathcal{G}(\Omega^\infty)$ is a viscosity subsolution (resp. supersolution) of* (3.16) *in Ω^∞ if for all test function $\phi \in \mathcal{G}(\Omega^\infty) \cap \mathcal{C}^\infty(\Omega^\infty)$ and for all points $\hat{\mathbf{x}} \in \Omega^\infty$ such that $(v^* - \phi)$ has a* global *maximum on Ω^∞ at $\hat{\mathbf{x}}$ and $v^*(\hat{\mathbf{x}}) = \phi(\hat{\mathbf{x}})$ (resp. $(v_* - \phi)$ has a* global *minimum on Ω^∞ at $\hat{\mathbf{x}}$ and $v_*(\hat{\mathbf{x}}) = \phi(\hat{\mathbf{x}})$), we have*

$$
\begin{aligned}
&(F_{\Omega^\infty})_* \left(\hat{\mathbf{x}}, \phi(\hat{\mathbf{x}}), D\phi(\hat{\mathbf{x}}), D^2\phi(\hat{\mathbf{x}}), \mathcal{J}\phi(\hat{\mathbf{x}}), \mathcal{M}\phi(\hat{\mathbf{x}})\right) \leq 0, \qquad \text{(B.1)} \\
\big(\textit{resp.} \quad &(F_{\Omega^\infty})^* \left(\hat{\mathbf{x}}, \phi(\hat{\mathbf{x}}), D\phi(\hat{\mathbf{x}}), D^2\phi(\hat{\mathbf{x}}), \mathcal{J}\phi(\hat{\mathbf{x}}), \mathcal{M}\phi(\hat{\mathbf{x}})\right) \geq 0, \big)
\end{aligned}
$$

where the operator $F_{\Omega^\infty}(\cdot)$ is defined in (3.9)*. A locally bounded function $v \in \mathcal{G}(\Omega^\infty)$ is a viscosity solution in $\Omega_{in} \cup \Omega_{a_{\min}}$ if it is both a viscosity subsolution and a viscosity supersolution in $\Omega_{in} \cup \Omega_{a_{\min}}$.*

The second definition is similar to [56, Definition 9.6], [61, Definition 5.3], [6, Definition 1], [60, Definition 2.2], and [27, Definition 4.2], which it is presented in Definition B.2 below. We find that it is more convenient to use this definition to prove a comparison principle.

Definition B.2 (Viscosity solution of equation (3.16))**.** *A locally bounded function $v \in \mathcal{G}(\Omega^\infty)$ is a viscosity subsolution (resp. supersolution) of* (3.16) *in Ω^∞ if for all test function $\phi \in \mathcal{G}(\Omega^\infty) \cap \mathcal{C}^\infty(\Omega^\infty)$ and for all points $\hat{\mathbf{x}} \in \Omega^\infty$ such that $(v^* - \phi)$ has a* local *maximum on Ω^∞ at $\hat{\mathbf{x}}$ and $v^*(\hat{\mathbf{x}}) = \phi(\hat{\mathbf{x}})$ (resp. $(v_* - \phi)$ has a* local *minimum on Ω^∞ at $\hat{\mathbf{x}}$ and $v_*(\hat{\mathbf{x}}) = \phi(\hat{\mathbf{x}})$), we have*

$$
\begin{aligned}
&(F_{\Omega^\infty})_* \left(\hat{\mathbf{x}}, \phi(\hat{\mathbf{x}}), D\phi(\hat{\mathbf{x}}), D^2\phi(\hat{\mathbf{x}}), \mathcal{J}v^*(\hat{\mathbf{x}}), \mathcal{M}v^*(\hat{\mathbf{x}})\right) \leq 0, \qquad \text{(B.2)} \\
\big(\textit{resp.} \quad &(F_{\Omega^\infty})^* \left(\hat{\mathbf{x}}, \phi(\hat{\mathbf{x}}), D\phi(\hat{\mathbf{x}}), D^2\phi(\hat{\mathbf{x}}), \mathcal{J}v_*(\hat{\mathbf{x}}), \mathcal{M}v_*(\hat{\mathbf{x}})\right) \geq 0, \big)
\end{aligned}
$$

where the operator $F_{\Omega^\infty}(\cdot)$ is defined in (3.9)*. A locally bounded function $v \in \mathcal{G}(\Omega^\infty)$ is a viscosity solution in $\Omega_{in} \cup \Omega_{a_{\min}}$ if it is both a viscosity subsolution and a viscosity supersolution in $\Omega_{in} \cup \Omega_{a_{\min}}$.*

Proposition B.1. *For the impulse control problem stated in Definition 3.1, Definition B.2 and Definition B.1 are equivalent.*

Proof. For a fixed $\mathbf{x} \in \Omega^\infty$, and $\delta > 0$, we define $\overline{B}_\delta(\mathbf{x}) = \{\mathbf{y} \in \Omega^\infty : |\mathbf{x} - \mathbf{y}| \leq \delta\}$.
Definition B.2 ⇒ Definition B.1: Since the jump operator $\mathcal{J}$ and intervention operator $\mathcal{M}$ are non-decreasing, it is straightforward to prove this part using the ellipticity of $F_{\Omega^\infty}(\cdot)$.
Definition B.1 ⇒ Definition B.2: In the below, we prove the "subsolution" case of this direction of implication. (The "supersolution" case can be handled similarly, and hence is omitted for brevity.) Specifically, assume that we are given (i) v as a viscosity subsolution in the sense of Definition B.1; and (ii) an arbitrary test function $\phi \in \mathcal{G}(\Omega^\infty) \cap \mathcal{C}^\infty(\Omega^\infty)$ such that $(v^* - \phi)$ has a local maximum at a point $\hat{\mathbf{x}} \in \overline{B}_\delta(\hat{\mathbf{x}}) \subset \Omega^\infty$ for some $\delta > 0$, and that $v^*(\hat{\mathbf{x}}) = \phi(\hat{\mathbf{x}})$. We now show that the inequality (B.2) holds.

Since $v^*(\mathbf{x})$ is upper semi-continuous, there exists $\phi' \in \mathcal{G}(\Omega^\infty) \cap \mathcal{C}^\infty(\Omega^\infty)$ such that, for any $\epsilon > 0$, we have $v^*(\mathbf{x}) \leq \phi'(\mathbf{x}) \leq v^*(\mathbf{x}) + \epsilon$, $\forall\ \mathbf{x} \in \Omega^\infty$. Let us consider a smooth cut-off function $\zeta(\mathbf{x})$ such that

$$
0 \leq \zeta(\mathbf{x}) \leq 1;\ \zeta(\mathbf{x}) \equiv 1\ \forall \mathbf{x} \in \overline{B}_{\delta/2}(\hat{\mathbf{x}});\ \zeta(\mathbf{x}) \equiv 0\ \ \forall \mathbf{x} \in \left\{\Omega^\infty \setminus \overline{B}_\delta(\hat{\mathbf{x}})\right\}.
$$

We then define a new function $\varphi(\mathbf{x}) := \zeta(\mathbf{x})\phi(\mathbf{x}) + (1-\zeta(\mathbf{x}))\,\phi'(\mathbf{x})$, $\mathbf{x} \in \Omega^\infty$. By construction of $\varphi(\mathbf{x})$, it follows that $\varphi \in \mathcal{G}(\Omega^\infty) \cap \mathcal{C}^\infty(\Omega^\infty)$ and

$$v^*(\mathbf{x}) \leq \varphi(\mathbf{x}) \leq v^*(\mathbf{x}) + \epsilon, \quad \forall\, \mathbf{x} \in \Omega^\infty. \tag{B.3}$$

We also have $v^*(\hat{\mathbf{x}}) = \varphi(\hat{\mathbf{x}})$, since $v^*(\hat{\mathbf{x}}) = \phi(\hat{\mathbf{x}})$ (by assumptions) and $\varphi(\hat{\mathbf{x}}) = \phi(\hat{\mathbf{x}})$ by construction of $\varphi(\mathbf{x})$. Following (B.3), we can conclude that $(v^* - \varphi)(\mathbf{x})$ has a global maximum on Ω^∞ at $\hat{\mathbf{x}}$ and $v^*(\hat{\mathbf{x}}) = \varphi(\hat{\mathbf{x}})$.

Since v is a viscosity subsolution in the sense of Definition B.1, using $\varphi(\mathbf{x})$ as the test function in (B.1), we arrive at (noting that $\varphi(\hat{\mathbf{x}}) = \phi(\hat{\mathbf{x}})$, $D\varphi(\hat{\mathbf{x}}) = D\phi(\hat{\mathbf{x}})$, $D^2\varphi(\hat{\mathbf{x}}) = D^2\phi(\hat{\mathbf{x}})$)

$$(F_{\Omega^\infty})_*(\hat{\mathbf{x}}, \phi(\hat{\mathbf{x}}), D\phi(\hat{\mathbf{x}}), D^2\phi(\hat{\mathbf{x}}), \mathcal{J}\varphi(\hat{\mathbf{x}}), \mathcal{M}\varphi(\hat{\mathbf{x}})) \leq 0. \tag{B.4}$$

Using (B.4), we will derive (B.2) case by case, depending where $\overline{B}_\delta(\hat{\mathbf{x}})$ is in Ω^∞.

- We first consider $\overline{B}_\delta(\hat{\mathbf{x}}) \subset \Omega_{\text{in}}$. By definition of $F_{\Omega^\infty}(\cdot)$ in (3.9), (B.4) becomes

$$\min\left[\phi_\tau(\hat{\mathbf{x}}) - \mathcal{L}\phi(\hat{\mathbf{x}}) - \mathcal{J}\varphi(\hat{\mathbf{x}}) - \sup_{\hat{\gamma}\in[0,C_r]} \hat{\gamma}\left(1 - e^{-w}\phi_w(\hat{\mathbf{x}}) - \phi_a(\hat{\mathbf{x}})\right), \phi(\hat{\mathbf{x}}) - \sup_{\gamma\in[0,a]} \mathcal{M}(\gamma)\varphi(\hat{\mathbf{x}})\right] \leq 0.$$

If the first argument in the above min operator is less than 0, using (B.3), we have that

$$\begin{aligned}
\phi_\tau(\hat{\mathbf{x}}) - \mathcal{L}\phi(\hat{\mathbf{x}}) - \sup_{\hat{\gamma}\in[0,C_r]} \hat{\gamma}\left(1 - e^{-w}\phi_w(\hat{\mathbf{x}}) - \phi_a(\hat{\mathbf{x}})\right) &\leq \lambda \int_{-\infty}^{\infty} \varphi(w+y, a, \tau)\; b(y)\; dy \\
&\leq \lambda \int_{-\infty}^{\infty} (v^*(w+y, a, \tau) + \epsilon)\; b(y)\; dy \\
&= \mathcal{J}v^*(\hat{\mathbf{x}}) + \lambda\epsilon.
\end{aligned} \tag{B.5}$$

Otherwise, if the second argument in the above min operator is less than 0, using (B.3) again gives

$$\begin{aligned}
\phi(\hat{\mathbf{x}}) &\leq \sup_{\gamma\in[0,a]} \left[\varphi\left(\ln(\max(e^w - \gamma, e^{w_{-\infty}})), a-\gamma, \tau\right) + (1-\mu)\gamma - c\right] \\
&\leq \sup_{\gamma\in[0,a]} \left[v^*\left(\ln(\max(e^w - \gamma, e^{w_{-\infty}})), a-\gamma, \tau\right) + \epsilon + (1-\mu)\gamma - c\right] \\
&= \sup_{\gamma\in[0,a]} \mathcal{M}(\gamma)v^*(\hat{\mathbf{x}}) + \epsilon.
\end{aligned} \tag{B.6}$$

Combining these two cases (B.5) and (B.6), and letting $\epsilon \to 0$, we have that

$$\min\left[\phi_\tau(\hat{\mathbf{x}}) - \mathcal{L}\phi(\hat{\mathbf{x}}) - \mathcal{J}v^*(\hat{\mathbf{x}}) - \sup_{\hat{\gamma}\in[0,C_r]} \hat{\gamma}\left(1 - e^{-w}\phi_w(\hat{\mathbf{x}}) - \phi_a(\hat{\mathbf{x}})\right), \phi(\hat{\mathbf{x}}) - \sup_{\gamma\in[0,a]} \mathcal{M}(\gamma)v^*(\hat{\mathbf{x}})\right] \leq 0,$$

which implies that

$$(F_{\Omega^\infty})_*(\hat{\mathbf{x}}, \phi(\hat{\mathbf{x}}), D\phi(\hat{\mathbf{x}}), D^2\phi(\hat{\mathbf{x}}), \mathcal{J}v^*(\hat{\mathbf{x}}), \mathcal{M}v^*(\hat{\mathbf{x}})) \leq 0. \tag{B.7}$$

- The other cases when $\overline{B}_\delta(\hat{\mathbf{x}}) \subset \Omega^\infty_{\tau_0}$, $\Omega^\infty_{w_{\min}}$, $\Omega^\infty_{wa_{\min}}$, $\Omega^\infty_{w_{\max}}$, or $\Omega_{a_{\min}}$ can be treated similarly.

- We then consider a special case when $\overline{B}_\delta(\hat{\mathbf{x}}) \subset \Omega_{\text{in}} \cup \Omega^\infty_{w_{\min}}$ and $\hat{\mathbf{x}} \in \{w_{\min}\} \times (a_{\min}, a_{\max}] \times (0, T]$. By definition of $F_{\Omega^\infty}(\cdot)$ in (3.9), (B.4) becomes

$$\min\left[F_{w_{\min}}(\hat{\mathbf{x}}, \phi(\hat{\mathbf{x}}), D\phi(\hat{\mathbf{x}}), \mathcal{M}\varphi(\hat{\mathbf{x}})),\; F_{\text{in}}(\hat{\mathbf{x}}, \phi(\hat{\mathbf{x}}), D\phi(\hat{\mathbf{x}}), D^2\phi(\hat{\mathbf{x}}), \mathcal{J}\varphi(\hat{\mathbf{x}}), \mathcal{M}\varphi(\hat{\mathbf{x}}))\right] \leq 0.$$

  Using the technique in (B.5) and (B.6), we can derive (B.7). All the other cases can be treated similarly.

Finally, we can conclude that v is a viscosity subsolution in the sense of Definition B.2. □

To facilitate our proof of a strong comparison principle in $\Omega_{\text{in}} \cup \Omega_{a_{\min}}$, following [6][Appendix A] and [5, 61, 65], in Definition B.3 below, we rewrite Definition B.2 specifically for the sub-domains $\Omega_{\text{in}} \cup \Omega_{a_{\min}}$, without using the envelopes $(F_{\Omega^\infty})_*$ and $(F_{\Omega^\infty})^*$. From the definition of the operator F_{Ω^∞}, we can deal with the lim inf and lim sup operators in $\Omega_{\text{in}} \cup \Omega_{a_{\min}}$, which yields the following definition of viscosity solution.

Definition B.3 (Viscosity solution of equation (3.16))**.** *A locally bounded function $v \in \mathcal{G}(\Omega^\infty)$ is a viscosity subsolution (resp. supersolution) of* (3.16) *in $\Omega_{in} \cup \Omega_{a_{\min}}$ if for all test functions $\phi \in \mathcal{G}(\Omega^\infty) \cap \mathcal{C}^\infty(\Omega^\infty)$ and for all points $\hat{\mathbf{x}} \in \Omega_{in} \cup \Omega_{a_{\min}}$ such that $(v^* - \phi)$ has a* local *maximum on $\Omega_{in} \cup \Omega_{a_{\min}}$ at $\hat{\mathbf{x}}$ and $v^*(\hat{\mathbf{x}}) = \phi(\hat{\mathbf{x}})$ (resp. $(v_* - \phi)$ has a* local *minimum on $\Omega_{in} \cup \Omega_{a_{\min}}$ at $\hat{\mathbf{x}}$ and $v_*(\hat{\mathbf{x}}) = \phi(\hat{\mathbf{x}})$), we have*

$$F_{\Omega^\infty}\left(\hat{\mathbf{x}}, \phi(\hat{\mathbf{x}}), D\phi(\hat{\mathbf{x}}), D^2\phi(\hat{\mathbf{x}}), \mathcal{J}v^*(\hat{\mathbf{x}}), \mathcal{M}v^*(\hat{\mathbf{x}})\right) \;\leq\; 0, \tag{B.8}$$

$$\Big(\textit{resp.} \quad F_{\Omega^\infty}\left(\hat{\mathbf{x}}, \phi(\hat{\mathbf{x}}), D\phi(\hat{\mathbf{x}}), D^2\phi(\hat{\mathbf{x}}), \mathcal{J}v_*(\hat{\mathbf{x}}), \mathcal{M}v_*(\hat{\mathbf{x}})\right) \;\geq\; 0, \quad \Big)$$

where the operator $F_{\Omega^\infty}(\cdot)$ is defined in (3.9)*. A locally bounded function $v \in \mathcal{G}(\Omega^\infty)$ is a viscosity solution in $\Omega_{in} \cup \Omega_{a_{\min}}$ if it is both a viscosity subsolution and a viscosity supersolution in $\Omega_{in} \cup \Omega_{a_{\min}}$.*

It is straightforward to show that a viscosity solution in $\Omega_{\text{in}} \cup \Omega_{a_{\min}}$ in the sense of Definition B.2 is a viscosity solution in $\Omega_{\text{in}} \cup \Omega_{a_{\min}}$ in the sense of Definition B.3. We will use Definition B.3 to prove a strong comparison principle in $\Omega_{\text{in}} \cup \Omega_{a_{\min}}$.

B.2 A strong comparison principle

Next, we follow [61, Lemma 5.10] to introduce a lemma.

Lemma B.1. *For the impulse control problem* (3.1)*, there exists a function $q \in \mathcal{G}(\Omega^\infty) \cap \mathcal{C}^\infty(\Omega^\infty)$ and a positive function $k : \Omega^\infty \to \mathbb{R}$ such that*

$$F_{\Omega^\infty}\left(\mathbf{x}, q(\mathbf{x}), Dq(\mathbf{x}), D^2q(\mathbf{x}), \mathcal{J}q(\mathbf{x}), \mathcal{M}q(\mathbf{x})\right) \;\geq\; k, \quad \mathbf{x} \in \Omega_{in} \cup \Omega_{a_{\min}}. \tag{B.9}$$

Then, for any viscosity supersolution v in the sense of Definition B.3 in $\Omega_{in} \cup \Omega_{a_{\min}}$, $v_m := (1 - \frac{1}{m})v + \frac{1}{m}q$, where $m \geq 1$, is a viscosity supersolution in the sense of Definition B.3 of

$$F_{\Omega^\infty}\left(\mathbf{x}, v(\mathbf{x}), Dv(\mathbf{x}), D^2v(\mathbf{x}), \mathcal{J}v(\mathbf{x}), \mathcal{M}v(\mathbf{x})\right) - k/m \;=\; 0, \quad \mathbf{x} \in \Omega_{in} \cup \Omega_{a_{\min}}. \tag{B.10}$$

A proof of the above lemma is straightforward, and hence omitted for brevity. For example, we can define a smooth perturbation function $q(\mathbf{x}) = a + c/r$ in Ω^∞, with c be the positive fixed cost, and then show that

$$F_{\Omega^\infty}\left(\mathbf{x}, q(\mathbf{x}), Dq(\mathbf{x}), D^2q(\mathbf{x}), \mathcal{J}q(\mathbf{x}), \mathcal{M}q(\mathbf{x})\right) \;\geq\; c, \quad \mathbf{x} \in \Omega_{\text{in}} \cup \Omega_{a_{\min}}.$$

Now we can proceed to proving a strong comparison principle in $\Omega_{\text{in}} \cup \Omega_{a_{\min}}$.

Theorem B.1. *Suppose that (i) a locally bounded and u.s.c. function $u : \Omega^\infty \to \mathbb{R}$ is a viscosity subsolution in the sense of Definition B.3 in $\Omega_{in} \cup \Omega_{a_{\min}}$, and (ii) a locally bounded and l.s.c. function $v : \Omega^\infty \to \mathbb{R}$ is a viscosity supersolution in the sense of Definition B.3 in $\Omega_{in} \cup \Omega_{a_{\min}}$, such that*

$$u(\mathbf{x}) \;\leq\; v(\mathbf{x}), \quad \forall\, \mathbf{x} \in \Omega^\infty_{out} \tag{B.11}$$

$$u(\mathbf{x}) := \limsup_{\substack{\mathbf{y} \to \mathbf{x} \\ \mathbf{y} \in \Omega_{in} \cup \Omega_{a_{\min}}}} u(\mathbf{y}) \;\leq\; v(\mathbf{x}) := \liminf_{\substack{\mathbf{y} \to \mathbf{x} \\ \mathbf{y} \in \Omega_{in} \cup \Omega_{a_{\min}}}} v(\mathbf{y}), \quad \forall\, \mathbf{x} \in \Omega^{in}_{\tau_0}, \tag{B.12}$$

where $\Omega^\infty_{out} := \{\mathbb{R} \setminus [w_{\min}, w_{\max}]\} \times [a_{\min}, a_{\max}] \times (0, T]$ and $\Omega^{in}_{\tau_0} := [w_{\min}, w_{\max}] \times [a_{\min}, a_{\max}] \times \{0\}$. Then $u \leq v$ in $\Omega_{in} \cup \Omega_{a_{\min}}$.

Proof. Following [65], we (re)define u and w for $\mathbf{x} \in \{w_{\min}, w_{\max}\} \times [a_{\min}, a_{\max}] \times (0, T]$ by

$$u(\mathbf{x}) = \limsup_{\substack{\mathbf{y} \to \mathbf{x} \\ \mathbf{y} \in \Omega_{\text{in}} \cup \Omega_{a_{\min}}}} u(\mathbf{y}) \quad \text{and} \quad v(\mathbf{x}) = \liminf_{\substack{\mathbf{y} \to \mathbf{x} \\ \mathbf{y} \in \Omega_{\text{in}} \cup \Omega_{a_{\min}}}} v(\mathbf{y}). \tag{B.13}$$

From (B.13), we have that u is u.s.c. on $\overline{\Omega}_{\text{in}}$ and v is l.s.c. on $\overline{\Omega}_{\text{in}}$, where $\overline{\Omega}_{\text{in}}$ is the closure of Ω_{in}, and also the closure of $\Omega_{\text{in}} \cup \Omega_{a_{\min}}$. Let q as given in Lemma B.1, and $v_m := (1 - \frac{1}{m})v + \frac{1}{m}q$ for all $m \in \{1, 2, \ldots\}$. Note that when we impose the operators $\mathcal{J}$ and $\mathcal{M}$ on u and v_m for any $\mathbf{x} \in \Omega_{\text{in}} \cup \Omega_{a_{\min}}$, we need to use information from $\Omega^\infty_{\text{out}}$. Using the condition (B.11), without loss of generality, we set $v \leq q$ in $\Omega^\infty_{\text{out}}$, which implies $u \leq v_m$ in these areas.

It is sufficient to prove that $u - v_m \leq 0$ for sufficiently large m. Let m be fixed for the moment. To prove by contradiction, let us firstly assume $Q := \sup_{\mathbf{x} \in \overline{\Omega}_{\text{in}}} [u(\mathbf{x}) - v_m(\mathbf{x})] > 0$. Denote $Q = u(\bar{\mathbf{x}}) - v_m(\bar{\mathbf{x}})$ with $\bar{\mathbf{x}} := (\bar{w}, \bar{a}, \bar{\tau})$. If $\bar{\mathbf{x}} \in \Omega^{\text{in}}_{\tau_0}$, then it contradicts with the condition (B.12).

- Now we consider the supremum Q is approximated from within the sub-domain Ω_{in}, i.e. $\bar{\mathbf{x}}$ is contained in some open subset $G \subset \Omega_{\text{in}}$ with compact closure $\overline{G}$. For any two points $\mathbf{x} := (w_x, a_x, \tau_x) \in \overline{G}$ and $\mathbf{y} := (w_y, a_y, \tau_y) \in \overline{G}$, we define a test function $\varphi_\varepsilon(\mathbf{x}, \mathbf{y})$, for any $\varepsilon > 0$, such that

$$\varphi_\varepsilon(\mathbf{x}, \mathbf{y}) = \frac{1}{2\varepsilon}\big|\mathbf{x} - \mathbf{y}\big|^2 := \frac{1}{2\varepsilon}\left[(w_x - w_y)^2 + (a_x - a_y)^2 + (\tau_x - \tau_y)^2\right],$$

and then we define

$$Q_\varepsilon = \sup_{(\mathbf{x},\mathbf{y}) \in \overline{G} \times \overline{G}} \left[u(\mathbf{x}) - v_m(\mathbf{y}) - \varphi_\varepsilon(\mathbf{x}, \mathbf{y})\right].$$

By the definition of u and v_m, the maximum must be attained on the compact set $\overline{G} \times \overline{G}$ (independent of ε). Choose a point $(\mathbf{x}_\varepsilon, \mathbf{y}_\varepsilon) \in \overline{G} \times \overline{G}$ where the maximum is attained. Following [22, Lemma 3.1], we obtain that $\frac{1}{2\varepsilon}|\mathbf{x}_\varepsilon - \mathbf{y}_\varepsilon|^2 \to 0$ as $\varepsilon \to 0$. Without loss of generality, we assume that we have chosen a sub-sequence of $\{\mathbf{x}_\varepsilon\}$ and $\{\mathbf{y}_\varepsilon\}$, converging to the same limit $\bar{\mathbf{x}}$ when $\varepsilon \to 0$. By the definition of φ_ε, We obtain that $Q_\varepsilon \to Q = u(\bar{\mathbf{x}}) - v_m(\bar{\mathbf{x}})$ for all limit points $\bar{\mathbf{x}}$ of $\{\mathbf{x}_\varepsilon\}$ and $\{\mathbf{y}_\varepsilon\}$. Let ε small enough such that $\mathbf{x}_\varepsilon, \mathbf{y}_\varepsilon \in \Omega_{\text{in}}$. To ease the notation, we rewrite $\mathcal{M}u(\mathbf{x}) \equiv \sup_{\gamma \in [0,a]} \mathcal{M}(\gamma)u(\mathbf{x})$ and rewrite the operator $F_{\text{in}}(\mathbf{x}, v)$ as

$$F_{\text{in}}(\mathbf{x}, v) \;\equiv\; \min\left[F(\mathbf{x}, v(\mathbf{x}), Dv(\mathbf{x}), D^2v(\mathbf{x}), \mathcal{J}v(\mathbf{x})), v(\mathbf{x}) - \mathcal{M}v(\mathbf{x})\right].$$

Using Lemma B.1, we know $v_m(\mathbf{y}_\varepsilon) - \mathcal{M}v_m(\mathbf{y}_\varepsilon) \geq k/m$.

  - If $u(\mathbf{x}_\varepsilon) - \mathcal{M}u(\mathbf{x}_\varepsilon) \leq 0$, by the definition of $\mathcal{M}$, we have for $\epsilon > 0$, there exists $\gamma_\epsilon \in [0, \bar{a}]$ such that

$$\begin{aligned} \mathcal{M}u(\bar{\mathbf{x}}) &\leq u\left(\ln(\max(e^{\bar{w}} - \gamma_\epsilon, e^{w_{-\infty}})), \bar{a} - \gamma_\epsilon, \bar{\tau}\right) + (1-\mu)\gamma_\epsilon - c + \epsilon, \\ \mathcal{M}v_m(\bar{\mathbf{x}}) &\geq v_m\left(\ln(\max(e^{\bar{w}} - \gamma_\epsilon, e^{w_{-\infty}})), \bar{a} - \gamma_\epsilon, \bar{\tau}\right) + (1-\mu)\gamma_\epsilon - c. \end{aligned} \tag{B.14}$$

    Note that $\mathcal{M}u$ is u.s.c. and $\mathcal{M}v_m$ is l.s.c. see [61, Lemma 4.3]. Thus, we derive that

$$\begin{aligned} Q &= \limsup_{\varepsilon \to 0}\left(u(\mathbf{x}_\varepsilon) - v_m(\mathbf{y}_\varepsilon)\right) \leq \limsup_{\varepsilon \to 0} \mathcal{M}u(\mathbf{x}_\varepsilon) - \liminf_{\varepsilon \to 0} \mathcal{M}v_m(\mathbf{y}_\varepsilon) - k/m \\ &\leq \mathcal{M}u(\bar{\mathbf{x}}) - \mathcal{M}v_m(\bar{\mathbf{x}}) - k/m \\ &\leq Q + \epsilon - k/m, \end{aligned} \tag{B.15}$$

    which is a contradiction for ϵ sufficiently small, and we use (B.14) in the last inequality.

  - If $u(\mathbf{x}_\varepsilon) - \mathcal{M}u(\mathbf{x}_\varepsilon) > 0$, we need apply Jenson-Ishii Lemma [22, Theorem 3.2].[6] To this end, following [22, Section 8], we make use of the parabolic semijets $\mathcal{P}^{2,\pm}_{\Omega}u(\mathbf{x}_\varepsilon)$ and their closures $\overline{\mathcal{P}}^{2,\pm}_{\Omega}u(\mathbf{x}_\varepsilon)$. Specifically, consider the maximum point $(\mathbf{x}_\varepsilon, \mathbf{y}_\varepsilon) \in \overline{G} \times \overline{G}$ of $(u - v_m - \varphi_\varepsilon)$, for any $\alpha > 0$, there exists $(D_{\mathbf{x}}\varphi_\varepsilon, X) \in \overline{\mathcal{P}}^{2,+}_{\Omega}u(\mathbf{x}_\varepsilon)$ and $(D_{\mathbf{y}}\varphi_\varepsilon, Y) \in \overline{\mathcal{P}}^{2,-}_{\Omega}v_m(\mathbf{y}_\varepsilon)$ such that

$$-3\alpha\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq 3\alpha\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}, \tag{B.16}$$

    and by definition of φ_ε, we obtain $D_{\mathbf{x}}\varphi_\varepsilon = -D_{\mathbf{y}}\varphi_\varepsilon = \varepsilon^{-1}(\mathbf{x}_\varepsilon - \mathbf{y}_\varepsilon)$.
    It remains to treat (using Lemma B.1 again)

$$\begin{aligned} F\left(\mathbf{x}_\varepsilon, u(\mathbf{x}_\varepsilon), \varepsilon^{-1}(\mathbf{x}_\varepsilon - \mathbf{y}_\varepsilon), X, \mathcal{J}u(\mathbf{x}_\varepsilon)\right) &\leq 0, \\ F\left(\mathbf{y}_\varepsilon, v_m(\mathbf{y}_\varepsilon), \varepsilon^{-1}(\mathbf{x}_\varepsilon - \mathbf{y}_\varepsilon), Y, \mathcal{J}v_m(\mathbf{y}_\varepsilon)\right) &\geq k/m. \end{aligned} \tag{B.17}$$

    Subtracting the above inequalities yields

$$\begin{aligned} k/m &\leq F\left(\mathbf{y}_\varepsilon, v_m(\mathbf{y}_\varepsilon), \varepsilon^{-1}(\mathbf{x}_\varepsilon - \mathbf{y}_\varepsilon), Y, \mathcal{J}v_m(\mathbf{y}_\varepsilon)\right) - F\left(\mathbf{x}_\varepsilon, u(\mathbf{x}_\varepsilon), \varepsilon^{-1}(\mathbf{x}_\varepsilon - \mathbf{y}_\varepsilon), X, \mathcal{J}u(\mathbf{x}_\varepsilon)\right) \\ &\leq (r + \lambda)\left(v_m(\mathbf{y}_\varepsilon) - u(\mathbf{x}_\varepsilon)\right) + \left(\mathcal{J}u(\mathbf{x}_\varepsilon) - \mathcal{J}v_m(\mathbf{y}_\varepsilon)\right), \end{aligned}$$

    where we cancel out the derivative terms. Next, letting $\varepsilon \to 0$ yields

$$\begin{aligned} k/m &\leq r\left(v_m(\bar{\mathbf{x}}) - u(\bar{\mathbf{x}})\right) + \lambda \int_{-\infty}^{\infty} \Big[\left(u(\bar{w} + y, \bar{a}, \bar{\tau}) - v_m(\bar{w} + y, \bar{w}, \bar{\tau})\right) \\ &\qquad - \left(u(\bar{\mathbf{x}}) - v_m(\bar{\mathbf{x}})\right)\Big]\, b(y)\; dy \\ &\leq -rQ, \end{aligned} \tag{B.18}$$

    which yields a contradiction.

[6] In [61], a non-local Jenson-Ishii Lemma (see Corollary 5.13) is applied there, due to the complex structure of the jump operator. For our case, the treatment of the linear jump operator can be referred to [2].

Similarly, we can construct a contradiction when the supremum Q is approximated from within the sub-domain $\Omega_{a_{\min}}$.

- Next we consider $\bar{\mathbf{x}} \in \{w_{\min}, w_{\max}\} \times [a_{\min}, a_{\max}] \times (0, T]$. From (B.13), there exists a sequence (denoted by $\{\mathbf{z}_i = (w_z^i, a_z^i, \tau_z^i); i = 1, 2, \ldots\}$) in some open subset of $\Omega_{\text{in}} \cup \Omega_{a_{\min}}$ (still denoted by $G \subset \Omega_{\text{in}} \cup \Omega_{a_{\min}}$ with compact closure $\overline{G}$) converging to $\bar{\mathbf{x}}$, such that $v_m(\mathbf{z}_i)$ tends to $v_m(\bar{\mathbf{x}})$ when i goes to infinity. We only consider the case when $G \subset \Omega_{\text{in}}$ below, and the other case when $G \subset \Omega_{a_{\min}}$ can be handled similarly. If $\bar{\mathbf{x}} \in \{w_{\max}\} \times [a_{\min}, a_{\max}] \times (0, T]$ (the case when $\bar{\mathbf{x}} \in \{w_{\min}\} \times [a_{\min}, a_{\max}] \times (0, T]$ can be handled similarly), we use the technique in [65] to handle the boundary area. Let $\varepsilon_i = |\mathbf{z}_i - \bar{\mathbf{x}}|$, and set

$$\varphi_i(\mathbf{x}, \mathbf{y}) = \frac{1}{2\varepsilon_i}\left|\mathbf{x} - \mathbf{y}\right|^2 + \frac{1}{4}\left(\frac{d(\mathbf{y})}{d(\mathbf{z}_i)} - 1\right)^4 + \frac{1}{4}\left|\mathbf{x} - \bar{\mathbf{x}}\right|^4,$$

where $d(\mathbf{y})$ denotes the distance from $\mathbf{y}$ to the boundary area, i.e. $d(\mathbf{y}) = w_{\max} - w_y$. Then we define

$$Q_i = \sup_{(\mathbf{x},\mathbf{y}) \in \overline{G} \times \overline{G}} \left[u(\mathbf{x}) - v_m(\mathbf{y}) - \varphi_i(\mathbf{x}, \mathbf{y})\right].$$

There exists $(\mathbf{x}_i, \mathbf{y}_i) \in \overline{G} \times \overline{G}$ such that $Q_i = u(\mathbf{x}_i) - v_m(\mathbf{y}_i) - \varphi_i(\mathbf{x}_i, \mathbf{y}_i)$. Denote $\mathbf{x}_i = (w_x^i, a_x^i, \tau_x^i)$ and $\mathbf{y}_i = (w_y^i, a_y^i, \tau_y^i)$. Moreover, there exists a subsequence of $(\mathbf{x}_i, \mathbf{y}_i)$, still denoted by $(\mathbf{x}_i, \mathbf{y}_i)$, converging to $(\mathbf{x}, \mathbf{y}) \in \overline{G} \times \overline{G}$. When i goes to infinity, we have

$$Q_i \geq u(\bar{\mathbf{x}}) - v_m(\mathbf{z}_i) - \frac{\varepsilon_i}{2} \to u(\bar{\mathbf{x}}) - v_m(\bar{\mathbf{x}}) = Q,$$

which yields $\frac{1}{2\varepsilon_i}\left|\mathbf{x}_i - \mathbf{y}_i\right|^2$ is bounded and $\mathbf{x} = \mathbf{y}$. On the other hand, we also have

$$0 \leq \limsup_{i\to\infty} \varphi_i(\mathbf{x}_i, \mathbf{y}_i) = \limsup_{i\to\infty} \left[u(\mathbf{x}_i) - v_m(\mathbf{y}_i) - Q_i\right] \leq u(\mathbf{x}) - v_m(\mathbf{x}) - Q \leq 0.$$

Thus, $\mathbf{x} = \bar{\mathbf{x}}$, $\frac{1}{2\varepsilon_i}\left|\mathbf{x}_i - \mathbf{y}_i\right|^2 \to 0$, and $d(\mathbf{y}_i) \geq d(\mathbf{z}_i)/2 > 0$ for i sufficiently large. In particular, $d(\mathbf{y}_i) = w_{\max} - w_y^i > 0$, and so $\mathbf{y}_i \in \Omega_{\text{in}}$. When i sufficiently large, we can also assume $\mathbf{x}_i, \mathbf{y}_i \in G$. The remaining proof is similar with the previous case when $\bar{\mathbf{x}}$ is attained in the sub-domain Ω_{in}. We present some details for the readers' convenience.

  - We can still have

$$\begin{aligned} Q &= \limsup_{i\to\infty} \left(u(\mathbf{x}_i) - v_m(\mathbf{y}_i)\right) \leq \limsup_{i\to\infty} \mathcal{M}u(\mathbf{x}_i) - \liminf_{i\to\infty} \mathcal{M}v_m(\mathbf{y}_i) - k/m \\ &\leq \mathcal{M}u(\bar{\mathbf{x}}) - \mathcal{M}v_m(\bar{\mathbf{x}}) - k/m, \end{aligned}$$

  which is a contradiction according to (B.15).

  - Now we can apply Jenson-Ishii Lemma. Consider the maximum point $(\mathbf{x}_i, \mathbf{y}_i) \in \overline{G} \times \overline{G}$ of $(u - v_m - \varphi_i)$, for any $\alpha > 0$, there exists $(D_{\mathbf{x}}\varphi_i, X) \in \overline{\mathcal{P}}_{\Omega}^{2,+} u(\mathbf{x}_i)$ and $(D_{\mathbf{y}}\varphi_i, Y) \in \overline{\mathcal{P}}_{\Omega}^{2,-} v_m(\mathbf{y}_i)$ such that (B.16) holds, and by definition of φ_i, we obtain

$$D_{\mathbf{x}}\varphi_i = \frac{(\mathbf{x}_i - \mathbf{y}_i)}{\varepsilon_i} + (\mathbf{x}_i - \bar{\mathbf{x}})^3 \quad \text{and} \quad D_{\mathbf{y}}\varphi_i = -\frac{(\mathbf{x}_i - \mathbf{y}_i)}{\varepsilon_i} - \frac{\mathbf{1}_w}{d(\mathbf{z}_i)}\left(\frac{d(\mathbf{y}_i)}{d(\mathbf{z}_i)} - 1\right)^3,$$

  with $\mathbf{1}_w := (1, 0, 0)$. Similarly with (B.17), we can have

$$\begin{aligned} F\left(\mathbf{x}_i, u(\mathbf{x}_i), \frac{(\mathbf{x}_i - \mathbf{y}_i)}{\varepsilon_i} + (\mathbf{x}_i - \bar{\mathbf{x}})^3, X, \mathcal{J}u(\mathbf{x}_i)\right) &\leq 0, \\ F\left(\mathbf{y}_i, v_m(\mathbf{y}_i), \frac{(\mathbf{x}_i - \mathbf{y}_i)}{\varepsilon_i} + \frac{\mathbf{1}_w}{d(\mathbf{z}_i)}\left(\frac{d(\mathbf{y}_i)}{d(\mathbf{z}_i)} - 1\right)^3, Y, \mathcal{J}v_m(\mathbf{y}_i)\right) &\geq k/m. \end{aligned}$$

Similarly with (B.18), subtracting the above inequalities, and letting $i \to \infty$ can derive

$$
\begin{aligned}
k/m \quad &\leq \quad (r+\lambda)\left(v_m(\mathbf{y}_i) - u(\mathbf{x}_i)\right) + \left(\mathcal{J}u(\mathbf{x}_i) - \mathcal{J}v_m(\mathbf{y}_i)\right) \\
&\quad + \left(r - \frac{\sigma^2}{2} - \lambda\kappa - \beta\right)\left[\left(w_x^i - \bar{w}\right)^3 - \frac{1}{w_{\max} - w_z^i}\left(\frac{w_{\max} - w_y^i}{w_{\max} - w_z^i} - 1\right)^3\right] \\
&\quad + \sup_{\hat{\gamma}\in[0,C_r]}\left|\hat{\gamma}\left(a_x^i - \bar{a}\right)^3 + \hat{\gamma}\left[\left(w_x^i - \bar{w}\right)^3 - \frac{1}{w_{\max} - w_z^i}\left(\frac{w_{\max} - w_y^i}{w_{\max} - w_z^i} - 1\right)^3\right]\right| \\
&\leq \quad (r+\lambda)\left(v_m(\bar{\mathbf{x}}) - u(\bar{\mathbf{x}})\right) + \left(\mathcal{J}u(\bar{\mathbf{x}}) - \mathcal{J}v_m(\bar{\mathbf{x}})\right) \quad (\text{since } i \to \infty) \\
&\leq \quad r\left(v_m(\bar{\mathbf{x}}) - u(\bar{\mathbf{x}})\right) + \lambda\int_{-\infty}^{\infty}\Big[\left(u(\bar{w}+y,\bar{a},\bar{\tau}) - v_m(\bar{w}+y,\bar{w},\bar{\tau})\right) \\
&\quad - \left(u(\bar{\mathbf{x}}) - v_m(\bar{\mathbf{x}})\right)\Big]\, b(y)\, dy \\
&\leq \quad -rQ,
\end{aligned}
$$

which yields a contradiction.

Combining all these cases concludes the proof. □

By combining the previous results, we finally obtain an characterization of the numerical solutions.

Corollary B.1. *For the functions $\overline{v}$ and $\underline{v}$, defined in* (5.65)*, we have $\overline{v} \leq \underline{v}$ in $\Omega_{in} \cup \Omega_{a_{\min}}$.*

Proof. In the proof of Theorem 5.1, we have shown that $\overline{v}$ (resp. $\underline{v}$) is a viscosity subsolution (resp. supersolution) of equation (3.16) in the sense of Definition B.1. By Proposition B.1, $\overline{v}$ (resp. $\underline{v}$) is also a viscosity subsolution (resp. supersolution) in the sense of Definition B.3. Here, the region of definition is $\Omega_{\text{in}} \cup \Omega_{a_{\min}}$.

To apply Theorem B.1, we only need to show that $\overline{v}(\mathbf{x})$ and $\underline{v}(\mathbf{x})$ satisfy condition (B.12) for all $\mathbf{x} \in \Omega_{\tau_0}^{\text{in}}$, noting condition (B.11) is trivially satisfied given the definition (5.65). We describe the main steps of this proof below.

- **Step 1** We prove a strong comparison result for an associated QVI. Note that for $w \in [w_{\min}, w_{\max}]$, $\max(e^w, (1-\mu)a - c) \wedge e^{w_\infty}$ trivially becomes $\max(e^w, (1-\mu)a - c)$. We ignore e^{w_∞} for brevity.
  - **Step 1.1** Recalling $\Omega_{\tau_0}^{\text{in}} := [w_{\min}, w_{\max}] \times [a_{\min}, a_{\max}] \times \{0\}$, we consider the QVI[7]

$$
\min\left[v - \max(e^w, (1-\mu)a - c),\ v - \sup_{\gamma\in[0,a]}\mathcal{M}(\gamma)v\right] \ = \ 0,\ \mathbf{x} \in \ \Omega_{\tau_0}^{\text{in}}. \tag{B.19}
$$

    We then define the viscosity solution of the QVI (B.19) in the sense of Definition B.3 below[8].

    Definition B.4 (Viscosity solution of (B.19))**.** *A locally bounded function $v \in \mathcal{G}(\Omega^\infty)$ is a viscosity subsolution (resp. supersolution) of* (B.19) *in $\Omega_{\tau_0}^{in}$ if for all test function $\phi \in \mathcal{G}(\Omega^\infty) \cap \mathcal{C}^\infty(\Omega^\infty)$ and for all points $\hat{\mathbf{x}} = (\hat{w}, \hat{a}, 0) \in \Omega_{\tau_0}^{in}$ such that $(v^* - \phi)$ has a* local *maximum on $\Omega_{\tau_0}^{in}$ at $\hat{\mathbf{x}}$ and $v^*(\hat{\mathbf{x}}) = \phi(\hat{\mathbf{x}})$ (resp. $(v_* - \phi)$ has a* local *minimum on $\Omega_{\tau_0}^{in}$ at $\hat{\mathbf{x}}$ and $v_*(\hat{\mathbf{x}}) = \phi(\hat{\mathbf{x}})$), we have*

$$
\min\left[\phi(\hat{\mathbf{x}}) - \max(e^{\hat{w}}, (1-\mu)\hat{a} - c),\ \phi(\hat{\mathbf{x}}) - \sup_{\gamma\in[0,a]}\mathcal{M}(\gamma)v^*(\hat{\mathbf{x}})\right] \quad \leq \quad 0,
$$

$$
\Big(\textit{resp.} \quad \min\left[\phi(\hat{\mathbf{x}}) - \max(e^{\hat{w}}, (1-\mu)\hat{a} - c),\ \phi(\hat{\mathbf{x}}) - \sup_{\gamma\in[0,a]}\mathcal{M}(\gamma)v_*(\hat{\mathbf{x}})\right] \quad \geq \quad 0.\Big)
$$

    A locally bounded function $v \in \mathcal{G}(\Omega^\infty)$ is a viscosity solution in $\Omega_{\tau_0}^{in}$ if it is both a viscosity subsolution and a viscosity supersolution in $\Omega_{\tau_0}^{in}$.

  - **Step 1.2** We prove a strong comparison principle for (B.19)[9].

    This can be done using similar arguments in Theorem B.1. (Also see [61, Theorem 5.9].) We can then conclude that, if $u(\mathbf{x})$ (resp. $v(\mathbf{x})$) is a viscosity subsolution (resp. supersolution) of equation (B.19) in the sense of Definition B.4, then $u(\mathbf{x}) \leq v(\mathbf{x})$ for all $\mathbf{x} \in \Omega_{\tau_0}^{\text{in}}$.

[7]When $a = a_{\min} = 0$, this QVI trivially becomes $v - e^w = 0$, which can be viewed as a special case.

[8]For the QVI (B.19), it is possible to fully remove the dependence on τ in the definition of viscosity solution. However, to facilitate the proofs for Step 2, we still require that $v \in \mathcal{G}(\Omega^\infty)$ in Definition B.4.

[9]Note that this result requires a similar condition to (B.11), which is satisfied by the function $\overline{v}$ and $\underline{v}$ in Step 3.

- **Step 2** We prove that $\overline{v}(\mathbf{x})$ and $\underline{v}(\mathbf{x})$, defined in (5.65), are viscosity subsolution and supersolution of (B.19) in the sense of Definition B.4, respectively. We will provide details for Step 2 below.
- **Step 3** By Step 2 and Step 3, we can conclude that $\overline{v}(\mathbf{x}) \leq \underline{v}(\mathbf{x})$ for all $\mathbf{x} \in \Omega^{\text{in}}_{\tau_0}$. This result shows that $\overline{v}\,(\mathbf{x})$ and $\underline{v}\,(\mathbf{x})$ satisfy condition (B.12) in Theorem B.1. Therefore, applying Theorem B.1 gives the desired result $\overline{v}\,(\mathbf{x}) \leq \underline{v}\,(\mathbf{x})$, $\forall \mathbf{x} \in \Omega_{\text{in}} \cup \Omega_{a_{\min}}$.

Below, we provide details for **Step 2**. By definition (5.65), $\overline{v}^*(\mathbf{x}) = \overline{v}(\mathbf{x})$ and $\underline{v}_*(\mathbf{x}) = \underline{v}(\mathbf{x})$, so we will work with $\overline{v}(\mathbf{x})$ and $\underline{v}(\mathbf{x})$ instead of the envelopes.

- **Step 2.1**: Using Theorem 5.1 and the equivalence between Definition B.1 and Definition B.2, we have $\overline{v}(\mathbf{x})$ (resp. $\underline{v}(\mathbf{x})$) is a viscosity subsolution (resp. supersolution) of equation (3.16) in the sense of Definition B.2 for all $\mathbf{x} \in \overline{\Omega}_{\text{in}} \subset \Omega^{\infty}$.
- **Step 2.2 ($\overline{v}(\mathbf{x})$ is a subsolution of** (B.19)**)**: Let $\phi \in \mathcal{G}(\Omega^{\infty}) \cap \mathcal{C}^{\infty}(\Omega^{\infty})$ and $\hat{\mathbf{x}} = (\hat{w}, \hat{a}, 0) \in \Omega^{\text{in}}_{\tau_0}$ be a point at which $(\overline{v} - \phi)(\hat{\mathbf{x}})$ is a local maximum and $\overline{v}(\hat{\mathbf{x}}) = \phi(\hat{\mathbf{x}})$. (We only consider the case when $\hat{\mathbf{x}} \in (w_{\min}, w_{\max}) \times (a_{\min}, a_{\max}] \times \{0\}$ below, and the other cases can be treated similarly.)

  Define $\varphi(w, a, \tau) := \phi(w, a, \tau) + C\tau$, where $C > 0$ is a constant to be chosen later. Since $\varphi(\mathbf{x}) \geq \phi(\mathbf{x})$ for all $\mathbf{x} \in \Omega^{\infty}$, and $\varphi(\mathbf{x}) = \phi(\mathbf{x})$ for all $\mathbf{x} \in \Omega^{\text{in}}_{\tau_0}$, it follows that $(\overline{v} - \varphi)(\hat{\mathbf{x}})$ is also a local maximum, and $\overline{v}(\hat{\mathbf{x}}) = \varphi(\hat{\mathbf{x}})$. Thus, by Step 2.1, we have

$$
\begin{aligned}
0 \;&\geq\; (F_{\Omega^{\infty}})_*\left(\hat{\mathbf{x}}, \varphi(\hat{\mathbf{x}}), D\varphi(\hat{\mathbf{x}}), D^2\varphi(\hat{\mathbf{x}}), \mathcal{J}\overline{v}(\hat{\mathbf{x}}), \mathcal{M}\overline{v}(\hat{\mathbf{x}})\right) \\
&=\; \min\Bigg[\phi_\tau(\hat{\mathbf{x}}) + C - \mathcal{L}\phi(\hat{\mathbf{x}}) - \mathcal{J}\overline{v}(\hat{\mathbf{x}}) - \sup_{\hat{\gamma} \in [0, C_r]} \hat{\gamma}\left(1 - e^{-\hat{w}}\phi_w(\hat{\mathbf{x}}) - \phi_a(\hat{\mathbf{x}})\right)\mathbf{1}_{\{\hat{a} > 0\}}, \\
&\qquad\qquad \phi(\hat{\mathbf{x}}) - \sup_{\gamma \in [0,a]} \mathcal{M}(\gamma)\overline{v}(\hat{\mathbf{x}}),\; \phi(\hat{\mathbf{x}}) - \max\left(e^{\hat{w}}, (1-\mu)\hat{a} - c\right)\Bigg].
\end{aligned}
$$

  By choosing C large enough, we have

$$
\min\left[\phi(\hat{\mathbf{x}}) - \max(e^{\hat{w}}, (1-\mu)\hat{a} - c),\; \phi(\hat{\mathbf{x}}) - \sup_{\gamma \in [0,a]} \mathcal{M}(\gamma)\overline{v}(\hat{\mathbf{x}})\right] \;\leq\; 0,
$$

  which implies that $\overline{v}(\mathbf{x})$ is a viscosity subsolution of (B.19) in the sense of Definition B.4 in $\Omega^{\text{in}}_{\tau_0}$.
- **Step 2.3 ($\underline{v}(\mathbf{x})$ is a supersolution of** (B.19)**)**: Similarly, let $\phi \in \mathcal{G}(\Omega^{\infty}) \cap \mathcal{C}^{\infty}(\Omega^{\infty})$ and $\hat{\mathbf{x}} = (\hat{w}, \hat{a}, 0) \in \Omega^{\text{in}}_{\tau_0}$ be a point at which $(\underline{v} - \phi)(\hat{\mathbf{x}})$ is a local minimum and $\underline{v}(\hat{\mathbf{x}}) = \phi(\hat{\mathbf{x}})$. (We only consider the case when $\hat{\mathbf{x}} \in (w_{\min}, w_{\max}) \times (a_{\min}, a_{\max}] \times \{0\}$ below, and the other cases can be treated similarly.)

  Define $\varphi(w, a, \tau) := \phi(w, a, \tau) - C\tau$, where $C > 0$ is a constant to be chosen later. Since $\varphi(\mathbf{x}) \leq \phi(\mathbf{x})$ for all $\mathbf{x} \in \Omega^{\infty}$, and $\varphi(\mathbf{x}) = \phi(\mathbf{x})$ for all $\mathbf{x} \in \Omega^{\text{in}}_{\tau_0}$, it follows that $(\underline{v} - \varphi)(\hat{\mathbf{x}})$ is also a local minimum, and $\underline{v}(\hat{\mathbf{x}}) = \varphi(\hat{\mathbf{x}})$. Thus, by Step 2.1, we have

$$
\begin{aligned}
0 \;&\leq\; (F_{\Omega^{\infty}})^*\left(\hat{\mathbf{x}}, \varphi(\hat{\mathbf{x}}), D\varphi(\hat{\mathbf{x}}), D^2\varphi(\hat{\mathbf{x}}), \mathcal{J}\underline{v}(\hat{\mathbf{x}}), \mathcal{M}\underline{v}(\hat{\mathbf{x}})\right) \\
&=\; \max\Bigg[\min\Bigg[\phi_\tau(\hat{\mathbf{x}}) - C - \mathcal{L}\phi(\hat{\mathbf{x}}) - \mathcal{J}\underline{v}(\hat{\mathbf{x}}) - \sup_{\hat{\gamma} \in [0, C_r]} \hat{\gamma}\left(1 - e^{-\hat{w}}\phi_w(\hat{\mathbf{x}}) - \phi_a(\hat{\mathbf{x}})\right)\mathbf{1}_{\{\hat{a} > 0\}}, \\
&\qquad\qquad \phi(\hat{\mathbf{x}}) - \sup_{\gamma \in [0,a]} \mathcal{M}(\gamma)\underline{v}(\hat{\mathbf{x}})\Bigg],\; \phi(\hat{\mathbf{x}}) - \max\left(e^{\hat{w}}, (1-\mu)\hat{a} - c\right)\Bigg].
\end{aligned}
$$

  By choosing C large enough, we have that

$$
\phi(\hat{\mathbf{x}}) - \max\left(e^{\hat{w}}, (1-\mu)\hat{a} - c\right) \;\geq\; 0. \tag{B.20}
$$

  By definition of $\underline{v}(\hat{\mathbf{x}})$, we have $\underline{v}(\hat{\mathbf{x}}) \leq \max\left(e^{\hat{w}}, (1-\mu)\hat{a} - c\right)$. By the definition of $\mathcal{M}$, we also have

$$
\sup_{\gamma \in [0,a]} \mathcal{M}(\gamma)\underline{v}(\hat{\mathbf{x}}) \leq \sup_{\gamma \in [0,a]} \mathcal{M}(\gamma) \max\left(e^{\hat{w}}, (1-\mu)\hat{a} - c\right) \leq \max\left(e^{\hat{w}}, (1-\mu)\hat{a} - c\right),
$$

  which yields that

$$
\phi(\hat{\mathbf{x}}) - \sup_{\gamma \in [0,a]} \mathcal{M}(\gamma)\underline{v}(\hat{\mathbf{x}}) \;\geq\; \phi - \max\left(e^{\hat{w}}, (1-\mu)\hat{a} - c\right) \;\geq\; 0. \tag{B.21}
$$

Combining (B.20) and (B.21), we have that

$$\min\left[\phi(\hat{\mathbf{x}}) - \max(e^{\hat{w}}, (1-\mu)\hat{a} - c), \phi(\hat{\mathbf{x}}) - \sup_{\gamma\in[0,a]} \mathcal{M}(\gamma)\underline{v}(\hat{\mathbf{x}})\right] \;\geq\; 0,$$

which implies that $\underline{v}(\mathbf{x})$ is a viscosity supersolution of (B.19) in the sense of Definition B.4 in $\Omega^{\text{in}}_{\tau_0}$.

□