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Constructing minimal telescopers for rational functions in three discrete variables

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A R T I C L E I N F O

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A B S T R A C T

We present a new algorithm for constructing minimal telescopers for rational functions in three discrete variables. This is the first discrete reduction-based algorithm that goes beyond the bivariate case. The termination of the algorithm is guaranteed by a known existence criterion of telescopers. Our approach has the important feature that it avoids the potentially costly computation of certificates. Computational experiments are also provided so as to illustrate the efficiency of our approach.

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1. Introduction

Creative telescoping [49,50] is a powerful tool used to find closed form solutions for definite sums and definite integrals. The method constructs a recurrence (resp. differential) equation satisfied by the definite sum (resp. integral) with closed form solutions over a specified domain resulting in formulas for the sum or integral. Methods for finding such closed form solutions are available for many special functions, with examples given in [2,44,4,9,12,30,19,33,6]. Even when no closed form exists the method of creative telescoping often remains useful. For example the resulting recurrence or differential equation enables one to determine asymptotic expansions and derive other interesting facts about the original sum or integral.

In the case of summation, specialized to the trivariate case, in order to compute a sum of the form

$$\sum_{y=a_1}^{b_1} \sum_{z=a_2}^{b_2} f(x,y,z),$$

the main task of creative telescoping consists in finding $c_0, \ldots, c_\rho$, rational functions (or polynomials) in x, not all zero, and two functions $g(x,y,z), h(x,y,z)$ in the same class of functions as $f(x,y,z)$ such that

$$c_0 f + c_1 \mathrm{S}_x(f) + \cdots + c_\rho \mathrm{S}_x^\rho(f) = (\mathrm{S}_y(g) - g) + (\mathrm{S}_z(h) - h), \tag{1.1}$$

where S_x, S_y and S_z denote shift operators in x, y and z, respectively. The number ρ may or may not be part of the input. If $c_0, c_1, \ldots, c_\rho$ and g, h are as above, then $L = c_0 + c_1 \mathrm{S}_x + \cdots + c_\rho \mathrm{S}_x^\rho$ is called a *telescoper* for f and (g,h) is a *certificate* for L.

The utility of creative telescoping is best demonstrated by examples. Suppose we want to find a closed form of the following multiple sum

$$\sum_{y=0}^{x} \sum_{z=0}^{x} f(x,y,z) \quad \text{with } f(x,y,z) = \frac{2y-x}{(x+y+1)(-2x+y-1)(x+z+1)}.$$

To this end, the method first constructs a telescoper $L = \mathrm{S}_x - 1$ for f and a corresponding certificate

$$(g,h) = \left(\frac{8x^2-2xy-y^2+19x-2y+11}{(x+y+1)(-2x+y-3)(-2x+y-2)(x+z+1)}, \frac{-x+2y-1}{(x+y+2)(-2x+y-3)(x+z+1)} \right)$$

such that

$$L(f) = f(x+1,y,z) - f(x,y,z) = g(x,y+1,z) - g(x,y,z) + h(x,y,z+1) - h(x,y,z).$$

Summing on both sides over y, z from zero to x, and applying the idea of telescoping to g for y and to h for z, respectively, yield

$$
\begin{aligned}
&\sum_{y=0}^{x}\sum_{z=0}^{x} f(x+1,y,z) - \sum_{y=0}^{x}\sum_{z=0}^{x} f(x,y,z) \\
&= \sum_{z=0}^{x}\big(g(x,x+1,z)-g(x,0,z)\big) + \sum_{y=0}^{x}\big(h(x,y,x+1)-h(x,y,0)\big).
\end{aligned}
$$

Employing the notation $F(x)=\sum_{y=0}^{x}\sum_{z=0}^{x} f(x,y,z)$, along with a range match-up, one obtains

$$
\begin{aligned}
F(x+1)-F(x) &= \sum_{z=0}^{x}\big(g(x,x+1,z)-g(x,0,z)+f(x+1,x+1,z)\big) \\
&\quad + \sum_{y=0}^{x}\big(h(x,y,x+1)-h(x,y,0)+f(x+1,y,x+1)\big) \\
&\quad + f(x+1,x+1,x+1) \\
&= \sum_{z=0}^{x}\frac{x+1}{(x+2)(2x+3)(x+z+1)(x+z+2)} \\
&\quad + \sum_{y=0}^{x}\frac{x-2y+1}{2(x+1)(2x+3)(x+y+2)(-2x+y-3)} - \frac{x+1}{(x+2)(2x+3)^2},
\end{aligned}
\tag{1.2}
$$

where the right-hand side merely involves single sums and thus the problem is now reduced to finding closed forms of these sums. Applying the method of creative telescoping (specialized to the bivariate case) again, one finds that the first single sum is equal to $1/(2(x+2)(2x+3))$, while the second sum admits a first-order linear recurrence equation, which yields the closed form $-1/(2(x+2)(2x+3)^2)$. A direct calculation confirms that the right-hand side of (1.2) collapses to zero after expansion, that is, $F(x+1)-F(x)=0$. Together with the initial value $F(0)=0$, one then concludes that $\sum_{y=0}^{x}\sum_{z=0}^{x} f(x,y,z)=0$.

Over the past two decades, a number of generalizations and refinements of creative telescoping have been developed. At the present time the reduction-based approach has gained high support as it is both efficient in practice and has the important feature of being able to find a telescoper for a given function without necessarily computing a corresponding certificate. This is desirable in a typical situation where only the telescoper is of interest and its size is much smaller than the size of the certificate. Even when a certificate is needed, the approach also allows one to express it as an unnormalized sum so that the summands are concatenated symbolically without actually calculating the sum. Such an expression can be more easily specialized at end points of the summation range than the expanded certificate, and thus turns out to be useful in many applications.

The reduction-based approach was first developed in the differential case for bivariate rational functions [14], and later generalized to rational functions in several variables [17], to hyperexponential functions [15], to algebraic functions [25] and to D-finite functions [22,34,16]. In the shift case a reduction-based approach was developed for hypergeometric terms [24,36] and to multiple binomial sums [18] (a subclass of the sums of hypergeometric terms).

In the case of discrete functions having more than two variables no complete reduction-based creative telescoping algorithm has been known so far. Having such an algorithm would allow us to tackle many multiple summations from applications more efficiently. However, it is quite challenging to develop an algorithm once for all. As a first step, in the present paper we address the most fundamental case, namely when f, g, h in (1.1) are all rational functions in x, y, z. This is also a natural follow up to the recent work [23,21,20] on the existence problem of telescopers for rational functions in three variables.

The basic idea of the general reduction-based approach, formulated for the shift trivariate rational case, is as follows. Let $\mathbb{K}$ be a field of characteristic zero. Assume that there is a $\mathbb{K}(x)$-linear map $\operatorname{red}(\cdot) : \mathbb{K}(x, y, z) \to \mathbb{K}(x, y, z)$ with the property that for all $f \in \mathbb{K}(x, y, z)$, there exist $g, h \in \mathbb{K}(x, y, z)$ such that $f - \operatorname{red}(f) = (\mathrm{S}_y(g) - g) + (\mathrm{S}_z(h) - h)$, that is, $f - \operatorname{red}(f)$ is summable with respect to y, z, and $\operatorname{red}(f)$ is minimal in certain sense. In other words, $\operatorname{red}(f)$ indicates the "minimum" adjustments needed for f to become summable with respect to y, z, which apparently excludes the most trivial case of $\operatorname{red}(f) = f$. Such a map is called a *reduction* with $\operatorname{red}(f)$ considered as a *remainder* of f with respect to the reduction $\operatorname{red}(\cdot)$. Then in order to find a telescoper for f, we can iteratively compute $\operatorname{red}(f), \operatorname{red}(\mathrm{S}_x(f)), \operatorname{red}(\mathrm{S}_x^2(f)), \ldots$ until we find a nontrivial linear dependence over $\mathbb{K}(x)$. Once we have such a dependence, say

$$c_0 \operatorname{red}(f) + \cdots + c_\rho \operatorname{red}(\mathrm{S}_x^\rho(f)) = 0$$

for $c_i \in \mathbb{K}(x)$ not all zero, then by linearity, $\operatorname{red}(c_0 f + \cdots + c_\rho \mathrm{S}_x^\rho(f)) = 0$, that is, $c_0 f + \cdots + c_\rho \mathrm{S}_x^\rho(f) = (\mathrm{S}_y(g) - g) + (\mathrm{S}_z(h) - h)$ for some $g, h \in \mathbb{K}(x, y, z)$. This yields a telescoper $c_0 + \cdots + c_\rho \mathrm{S}_x^\rho$ for f.

To guarantee the termination of the above process, one possible way is to show that, for every summable function f, we have $\operatorname{red}(f) = 0$. If this is the case and $L = c_0 + \cdots + c_\rho \mathrm{S}_x^\rho$ is a telescoper for f, then $L(f)$ is summable by the definition. So $\operatorname{red}(L(f)) = 0$, and again by the linearity, $\operatorname{red}(f), \ldots, \operatorname{red}(\mathrm{S}_x^\rho(f))$ are linearly dependent over $\mathbb{K}(x)$. This means that we will not miss any telescoper and that the method will terminate provided that a telescoper is known to exist. This approach was taken in [24]. It requires us to know exactly under what kind of conditions a telescoper exists, so-called the *existence problem of telescopers*, and, when these conditions are fulfilled, then it is guaranteed to find one of minimal order ρ. Such existence problems have been well studied in the case of bivariate hypergeometric terms [5] and more recently in the trivariate rational case [23,21,20].

A second, alternate way to ensure termination, used for example in [14,15], is to show that, for a given function f, the remainders $\operatorname{red}(f), \operatorname{red}(\mathrm{S}_x(f)), \operatorname{red}(\mathrm{S}_x^2(f)), \ldots$ form a finite-dimensional $\mathbb{K}(x)$-vector space. Then, as soon as ρ exceeds this finite dimension, one can be sure that a telescoper of order at most ρ will be found. This also implies that every bound for the dimension gives rise to an upper bound for the minimal order of telescopers. This approach provides an independent proof for the existence of a telescoper. However, since such an upper order bound is only of theoretical interest and will not affect the practical efficiency of the algorithms, in this paper we will confine ourselves with the first approach for termination and leave the second approach for future research.

Our starting point is thus to find a suitable reduction for trivariate rational functions. In particular we present a reduction $\mathrm{red}(\cdot)$ which satisfies the following properties: (i) $\mathrm{red}(f) = 0$ whenever $f \in \mathbb{K}(x, y, z)$ is summable and (ii) $\mathrm{red}(f)$ is minimal in certain sense. One issue with this reduction, similar to that encountered in the bivariate hypergeometric case [24], is the difficulty that $\mathrm{red}(\cdot)$ is not a $\mathbb{K}(x)$-linear map in general. To overcome this we follow the ideas of [24]. Namely, we introduce the idea of congruences modulo summable rational functions and show that $\mathrm{red}(\cdot)$ becomes $\mathbb{K}(x)$-linear when it is viewed as a residue class. Using the existence criterion of telescopers established in [23], we are then able to design a creative telescoping algorithm from $\mathrm{red}(\cdot)$ as described in the previous paragraphs.

The remainder of the paper proceeds as follows. The next section gives some preliminary materials needed for this paper, particularly a review of a reduction method due to Abramov. In Section 3 we extend Abramov's reduction method to the bivariate case by incorporating a primary reduction. In Section 4 we show that the reduction remainders introduced in the previous section are well-behaved with respect to taking linear combinations, followed in Section 5 by a new algorithm for constructing telescopers for trivariate rational functions based on the bivariate extension of Abramov's reduction method. In Section 6 we provide some experimental tests of our new algorithm. The paper ends with some topics for future research.

2. Preliminaries

Throughout the paper we let $\mathbb{K}$ denote a field of characteristic zero, with $\mathbb{F} = \mathbb{K}(x)$ and $\mathbb{F}(y, z)$ being the field of rational functions in y, z over $\mathbb{F}$. Choosing the pure lexicographic order $y \prec z$, we say that a polynomial in $\mathbb{F}[y, z]$ is *monic* if its highest term with respect to y, z has coefficient one. For a nonzero polynomial $p \in \mathbb{F}[y, z]$, its degree and leading coefficient with respect to the variable $v \in \{y, z\}$ are denoted by $\deg_v(p)$ and $\mathrm{lc}_v(p)$, respectively. We will follow the convention that $\deg_v(0) = -\infty$.

We let σ_y and σ_z be the automorphisms over $\mathbb{F}(y, z)$, which, for any $f \in \mathbb{F}(y, z)$, are defined by

$$\sigma_y(f(x, y, z)) = f(x, y + 1, z) \quad \text{and} \quad \sigma_z(f(x, y, z)) = f(x, y, z + 1).$$

Let $G = \langle \sigma_y, \sigma_z \rangle$ be the free abelian multiplicative group generated by σ_y, σ_z. The application of an element $\tau = \sigma_y^\alpha \sigma_z^\beta$ in G to a rational function $f \in \mathbb{F}(y, z)$ is defined as

$$\tau(f(x,y,z)) = \sigma_y^\alpha \sigma_z^\beta (f(x,y,z)) = f(x, y+\alpha, z+\beta).$$

For any $\tau \in G$, we say that a polynomial $p \in \mathbb{F}[y,z]$ is *τ-free* if $\gcd(p, \tau^\ell(p)) = 1$ for all nonzero $\ell \in \mathbb{Z}$. A rational function $f \in \mathbb{F}(y,z)$ is called *τ-summable* if $f = \tau(g) - g$ for some $g \in \mathbb{F}(y,z)$. The *τ-summability problem* is then to decide whether a given rational function in $\mathbb{F}(y,z)$ is τ-summable or not. Rather than merely giving a negative answer in case the function is not τ-summable, one could instead seek solutions for a more general problem, namely the *τ-decomposition problem*, with the intent to make the nonsummable part as small as possible. Precisely speaking, the τ-decomposition problem, for a given rational function $f \in \mathbb{F}(y,z)$, asks for an additive decomposition of the form $f = \tau(g) - g + r$, where $g, r \in \mathbb{F}(y,z)$ and r is minimal in certain sense such that f would be τ-summable if and only if $r = 0$. It is readily seen that any solution to the decomposition problem tackles the corresponding summability problem as well.

In the case where $\tau = \sigma_y$, the decomposition problem was first solved by Abramov in [1] with refined algorithms in [3,43,10,31,45]. All these algorithms can be viewed as discrete analogues of the Ostrogradsky-Hermite reduction for rational integration (and beyond). We refer to any of these algorithms restricted to the rational case as the Abramov reduction.

Theorem 2.1 *(Abramov reduction). Let f be a rational function in $\mathbb{F}(y,z)$. Then the Abramov reduction finds $g \in \mathbb{F}(y,z)$ and $a, b \in \mathbb{F}[y,z]$ with $\deg_y(a) < \deg_y(b)$ and b being σ_y-free such that*

$$f = \sigma_y(g) - g + \frac{a}{b}.$$

Moreover, if f admits such a decomposition then

- *f is σ_y-summable if and only if $a = 0$;*
- *b has the lowest possible degree in y when $\gcd(a,b) = 1$. That is, if there exist a second $g' \in \mathbb{F}(y,z)$ and $a', b' \in \mathbb{F}[y,z]$ such that $f = \sigma_y(g') - g' + a'/b'$, then $\deg_y(b') \geq \deg_y(b)$.*

In view of the above theorem, we introduce the following definition.

Definition 2.2. A rational function $a/b \in \mathbb{F}(y,z)$ with $a, b \in \mathbb{F}[y,z]$ and $b \neq 0$ is called a *σ_y-remainder* if $\deg_y(a) < \deg_y(b)$ and b is σ_y-free.

It is evident from Theorem 2.1 that any nonzero σ_y-remainder is not σ_y-summable.

Generalizing to the bivariate case, we consider the (σ_y, σ_z)-summability problem of deciding whether a given rational function $f \in \mathbb{F}(y,z)$ can be written in the form $f = \sigma_y(g) - g + \sigma_z(h) - h$ for $g, h \in \mathbb{F}(y,z)$. If such a form exists, we say that f is (σ_y, σ_z)-

summable, abbreviated as summable in certain instances. The (σ_y, σ_z)-decomposition problem is then to decompose a given rational function $f \in \mathbb{F}(y, z)$ into the form

$$f = \sigma_y(g) - g + \sigma_z(h) - h + r,$$

where $g, h, r \in \mathbb{F}(y, z)$ and r is minimal in certain sense. Moreover, f is (σ_y, σ_z)-summable if and only if $r = 0$.

Recall [7] that an irreducible polynomial $f \in \mathbb{F}[y, z]$ is called (y, z)-*integer linear* over the field $\mathbb{F}$ if it can be written in the form $f = p(\alpha y + \beta z)$ for a polynomial $p(Z) \in \mathbb{F}[Z]$ and integers $\alpha, \beta \in \mathbb{Z}$. One may assume without loss of generality that $\beta \geq 0$ and α, β are coprime. A polynomial in $\mathbb{F}[y, z]$ is called (y, z)-*integer linear* over $\mathbb{F}$ if all its irreducible factors are (y, z)-integer linear over $\mathbb{F}$ while a rational function in $\mathbb{F}(y, z)$ is called (y, z)-*integer linear* over $\mathbb{F}$ if its numerator and denominator are both (y, z)-integer linear over $\mathbb{F}$. For simplicity, we just say a rational function is (y, z)-integer linear over $\mathbb{F}$ of (α, β)-type if it is equal to $p(\alpha y + \beta z)$ for some $p(Z) \in \mathbb{F}(Z)$ and α, β are coprime integers with $\beta \geq 0$. Algorithms for determining integer linearity can be found in [7,41,32].

In the context of creative telescoping, we will also need to consider the variable x and the automorphism σ_x, which for every $f \in \mathbb{F}(y, z)$ maps $f(x, y, z)$ to $f(x + 1, y, z)$. Two polynomials $p, q \in \mathbb{K}[x, y, z]$ are called (x, y, z)-*shift equivalent*, denoted by $p \sim_{x,y,z} q$, if there exist three integers ℓ, m, n such that $p = \sigma_x^\ell \sigma_y^m \sigma_z^n(q)$. We generalize this notion to the domain $\mathbb{F}[y, z]$ by saying that two polynomials $p, q \in \mathbb{F}[y, z]$ are (x, y, z)-shift equivalent if $p = \sigma_x^\ell \sigma_y^m \sigma_z^n(q)$ for integers ℓ, m, n. When $\ell = 0$ then p is also called (y, z)-*shift equivalent* to q, denoted by $p \sim_{y,z} q$. Clearly, both $\sim_{x,y,z}$ and $\sim_{y,z}$ are equivalence relations.

Let $\mathbb{F}(y, z)[\mathrm{S}_x, \mathrm{S}_y, \mathrm{S}_z]$ be the ring of linear recurrence operators in x, y, z over $\mathbb{F}(y, z)$. Here $\mathrm{S}_x, \mathrm{S}_y, \mathrm{S}_z$ commute with each other, and $\mathrm{S}_v f = \sigma_v(f)\mathrm{S}_v$ for any $f \in \mathbb{F}(y, z)$ and $v \in \{x, y, z\}$. The application of an operator $P = \sum_{i,j,k \geq 0} p_{ijk}\, \mathrm{S}_x^i\, \mathrm{S}_y^j\, \mathrm{S}_z^k$ in $\mathbb{F}(y, z)[\mathrm{S}_x, \mathrm{S}_y, \mathrm{S}_z]$ to a rational function $f \in \mathbb{F}(y, z)$ is then defined as

$$P(f) = \sum_{i,j,k \geq 0} p_{ijk} \sigma_x^i \sigma_y^j \sigma_z^k(f).$$

Definition 2.3. Let f be a rational function in $\mathbb{F}(y, z)$. A nonzero linear recurrence operator $L \in \mathbb{F}[S_x]$ is called a *telescoper* for f if $L(f)$ is (σ_y, σ_z)-summable, or equivalently, if there exist rational functions $g, h \in \mathbb{F}(y, z)$ such that

$$L(f) = (S_y - 1)(g) + (S_z - 1)(h),$$

where 1 denotes the identity map of $\mathbb{F}(y, z)$. We call (g, h) a corresponding certificate for L. The *order* of a telescoper is defined to be its degree in S_x. A telescoper of minimal order for f is called a *minimal telescoper* for f.

3. Bivariate extension of the Abramov reduction

In this section, we demonstrate how to solve the bivariate decomposition problem (and thus also the bivariate summability problem) using the Abramov reduction. To this end, let us first recall some key results on the bivariate summability from [35].

Based on a theoretical criterion given in [26, Theorem 3.7], Hou and Wang [35] developed an algorithm for solving the (σ_y, σ_z)-summability problem. The proof found in [35, Lemma 3.1] contains a reduction algorithm with inputs and outputs specified as follows.

Primary reduction. Given a rational function $f \in \mathbb{F}(y, z)$, compute rational functions $g, h, r \in \mathbb{F}(y, z)$ such that

$$f = (\mathrm{S}_y - 1)(g) + (\mathrm{S}_z - 1)(h) + r \tag{3.1}$$

and r is of the form

$$r = \sum_{i=1}^{m} \sum_{j=1}^{n_i} \frac{a_{ij}}{b_{ij} d_i^j} \tag{3.2}$$

with $m, n_i \in \mathbb{N}$, $a_{ij}, d_i \in \mathbb{F}[y, z]$ and $b_{ij} \in \mathbb{F}[y]$ satisfying that

- $\deg_z(a_{ij}) < \deg_z(d_i)$,
- d_i is a monic irreducible factor of the denominator of r and of positive degree in z,
- $d_i \not\sim_{y,z} d_{i'}$ whenever $i \neq i'$ for $1 \leq i, i' \leq m$.

Let f be a rational function in $\mathbb{F}(y, z)$ and assume that applying the primary reduction to f yields (3.1). Deciding if f is (σ_y, σ_z)-summable then amounts to checking the summability of r. By [35, Lemma 3.2], this is equivalent to checking the summability of each simple fraction $a_{ij}/(b_{ij} d_i^j)$. Thus the bivariate summability problem for a general rational function is reduced to determining the summability of several simple fractions, which in turn can be addressed by the following.

Theorem 3.1 *([35, Theorem 3.3]). Let $f = a/(b\, d^j)$, where $a, d \in \mathbb{F}[y, z]$, $b \in \mathbb{F}[y]$, $j \in \mathbb{N} \setminus \{0\}$ with d irreducible and $0 \leq \deg_z(a) < \deg_z(d)$. Then f is (σ_y, σ_z)-summable if and only if*

(i) *d is (y, z)-integer linear over $\mathbb{F}$ of (α, β)-type,*
(ii) *there exists $q \in \mathbb{F}(y)[z]$ with $\deg_z(q) < \deg_z(d)$ so that*

$$\frac{a}{b} = \sigma_y^{\beta} \sigma_z^{-\alpha}(q) - q. \tag{3.3}$$

Since d is irreducible, the first condition is easily recognized by comparing coefficients once d is known. In [35, §4], the second condition is checked by finding a polynomial

$$
\begin{tikzcd}
R_1 \arrow[r, "\sigma_1"] \arrow[d, "\psi"'] \arrow[dr, phantom, "\circlearrowright"] & R_1 \arrow[d, "\psi"] \\
R_2 \arrow[r, "\sigma_2"'] & R_2
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{F}(y,z) \arrow[r, "\tau = \sigma_y^{\beta}\sigma_z^{-\alpha}"] \arrow[d, "\phi_{\alpha,\beta}"'] \arrow[dr, phantom, "\circlearrowright"] & \mathbb{F}(y,z) \arrow[d, "\phi_{\alpha,\beta}"] \\
\mathbb{F}(y,z) \arrow[r, "\sigma_y"'] & \mathbb{F}(y,z)
\end{tikzcd}
$$

Fig. 1. Commutative diagrams for difference homomorphisms/isomorphisms.

solution of a system of linear recurrence equations in one variable based on a universal denominator derived from the m-fold Gosper representation. Such a polynomial solution gives rise to a desired q for (3.3).

In the rest of this section, we show how to detect the second condition via the Abramov reduction, without solving any auxiliary recurrence equations. As a result, we obtain an additive decomposition of the given rational function in $\mathbb{F}(y,z)$, from which one can not only read off the (σ_y, σ_z)-summability, but also gather useful descriptions on the possible "minimal" nonsummable part. This lays the foundation of our new algorithm in Section 5.

Let R be a ring (resp. field) and $\sigma : R \to R$ be an automorphism of R. The pair (R, σ) is called a *difference ring (resp. field)*. An element $r \in R$ is called a *constant* of R with respect to σ if $\sigma(r) = r$. The set of all such constants forms a subring (resp. subfield) of R, called the *constant subring (resp. subfield)* of R with respect to σ. Let (R_1, σ_1) and (R_2, σ_2) be two difference rings. A homomorphism (resp. isomorphism) $\psi : R_1 \to R_2$ is called a *difference homomorphism (resp. isomorphism)* from (R_1, σ_1) to (R_2, σ_2) if $\sigma_2 \circ \psi = \psi \circ \sigma_1$, that is, the left diagram in Fig. 1 commutes. Two difference rings are then said to be *isomorphic* if there exists a difference isomorphism between them.

Let α, β be two integers with β nonzero. We define an $\mathbb{F}$-homomorphism $\phi_{\alpha,\beta} : \mathbb{F}(y,z) \to \mathbb{F}(y,z)$ by

$$\phi_{\alpha,\beta}(y) = \beta y \quad \text{and} \quad \phi_{\alpha,\beta}(z) = \beta^{-1} z - \alpha y.$$

It is readily seen that $\phi_{\alpha,\beta}$ is an $\mathbb{F}$-isomorphism with inverse $\phi_{\alpha,\beta}^{-1}$ given by

$$\phi_{\alpha,\beta}^{-1}(y) = \beta^{-1} y \quad \text{and} \quad \phi_{\alpha,\beta}^{-1}(z) = \beta z + \alpha y.$$

We call $\phi_{\alpha,\beta}$ the *map for (α, β)-shift reduction.*

Proposition 3.2. *Let $\alpha, \beta \in \mathbb{Z}$ with $\beta \neq 0$ and $\tau = \sigma_y^{\beta}\sigma_z^{-\alpha}$. Then $\phi_{\alpha,\beta}$ is a difference isomorphism from $(\mathbb{F}(y,z), \tau)$ to $(\mathbb{F}(y,z), \sigma_y)$.*

Proof. Since $\phi_{\alpha,\beta}$ is an $\mathbb{F}$-isomorphism, it remains to show that $\sigma_y \circ \phi_{\alpha,\beta} = \phi_{\alpha,\beta} \circ \tau$, namely the right diagram in Fig. 1 commutes. This is confirmed by the observation that

$$\sigma_y(\phi_{\alpha,\beta}(f(y,z))) = \sigma_y(f(\beta y, \beta^{-1} z - \alpha y)) = f(\beta y + \beta, \beta^{-1} z - \alpha y - \alpha)$$

and

$$\phi_{\alpha,\beta}(\tau(f(y,z))) = \phi_{\alpha,\beta}(f(y+\beta, z-\alpha)) = f(\beta y+\beta, \beta^{-1}z-\alpha y-\alpha)$$

for any $f \in \mathbb{F}(y,z)$. □

Corollary 3.3. *Let $f \in \mathbb{F}(y,z)$ and assume the conditions of Proposition 3.2. Then f is τ-summable if and only if $\phi_{\alpha,\beta}(f)$ is σ_y-summable.*

Proof. By Proposition 3.2, $\phi_{\alpha,\beta}$ is a difference isomorphism from $(\mathbb{F}(y,z), \tau)$ to $(\mathbb{F}(y,z), \sigma_y)$. It follows that

$$f = \tau(g) - g \quad \Longleftrightarrow \quad \phi_{\alpha,\beta}(f) = \phi_{\alpha,\beta}(\tau(g)-g) = \sigma_y(\phi_{\alpha,\beta}(g)) - \phi_{\alpha,\beta}(g)$$

for any $g \in \mathbb{F}(y,z)$. The assertion follows. □

The problem of deciding whether a rational function $f \in \mathbb{F}(y)[z]$ satisfies the equation (3.3), that is, the $\sigma_y^{\beta}\sigma_z^{-\alpha}$-summability problem for f, is then equivalent to the σ_y-summability problem for $\phi_{\alpha,\beta}(f)$. In fact, there is also a natural one-to-one correspondence between additive decompositions of f with respect to $\sigma_y^{\beta}\sigma_z^{-\alpha}$ and additive decompositions of $\phi_{\alpha,\beta}(f)$ with respect to σ_y. Together with Definition 2.2, this motivates us to introduce the notions of remainder fractions and remainders, in order to characterize nonsummable rational functions concretely.

Definition 3.4. A fraction $a/(b\,d^j)$ with $a, d \in \mathbb{F}[y,z]$, $b \in \mathbb{F}[y]$ and $j \in \mathbb{N} \setminus \{0\}$ is called a *remainder fraction* if

- $\deg_z(a) < \deg_z(d)$;
- d is monic, irreducible and of positive degree in z;
- $\phi_{\alpha,\beta}(a/b)$ is a σ_y-remainder in case d is (y,z)-integer linear over $\mathbb{F}$ of (α,β)-type.

Definition 3.5. Let f be a rational function in $\mathbb{F}(y,z)$. Then $r \in \mathbb{F}(y,z)$ is called a *(σ_y, σ_z)-remainder* of f if $f - r$ is (σ_y, σ_z)-summable and r can be written as

$$r = \sum_{i=1}^{m}\sum_{j=1}^{n_i} \frac{a_{ij}}{b_{ij}d_i^j}, \tag{3.4}$$

where $m, n_i \in \mathbb{N}$, $a_{ij}, d_i \in \mathbb{F}[y,z]$, $b_{ij} \in \mathbb{F}[y]$ with each $a_{ij}/(b_{ij}d_i^j)$ being a remainder fraction, d_i being a factor of the denominator of r, and $d_i \nsim_{y,z} d_{i'}$ whenever $i \neq i'$ and $1 \leq i, i' \leq m$. For brevity, we just say that r is a (σ_y, σ_z)-remainder if f is clear from the context. We refer to (3.4), along with the attached conditions, as the *remainder form* of r.

The uniqueness of partial fraction decompositions (in this case with respect to z) implies that the remainder form of a given (σ_y, σ_z)-remainder is unique up to reordering and multiplication by units of $\mathbb{F}$. Evidently, every single remainder fraction, or part of summands in (3.4), is a (σ_y, σ_z)-remainder. Remainders not only help us to recognize summability, but also describe the "minimum" gap between a given rational function and summable rational functions, as shown in the next two propositions.

Proposition 3.6. *Let $r \in \mathbb{F}(y,z)$ be a nonzero (σ_y, σ_z)-remainder with the form (3.4). Then each nonzero $a_{ij}/(b_{ij}d_i^j)$ for $1 \le i \le m$ and $1 \le j \le n_i$, as well as r itself, is not (σ_y, σ_z)-summable.*

Proof. Since r is a (σ_y, σ_z)-remainder, each $a_{ij}/(b_{ij}d_i^j)$ is a remainder fraction. For a particular nonzero $a_{ij}/(b_{ij}d_i^j)$, namely $a_{ij} \neq 0$, we claim that it is not (σ_y, σ_z)-summable. If d_i is not (y,z)-integer linear over $\mathbb{F}$, then the claim follows by Theorem 3.1. Otherwise, assume that d_i is (y,z)-integer linear over $\mathbb{F}$ of (α, β)-type. Since $a_{ij}/(b_{ij}d_i^j)$ is a remainder fraction, Definition 3.4 reads that $\phi_{\alpha,\beta}(a_{ij}/b_{ij})$ is a σ_y-remainder and thus is not σ_y-summable. By Corollary 3.3, a_{ij}/b_{ij} is not $\sigma_y^\beta \sigma_z^{-\alpha}$-summable. The claim is then again assured by Theorem 3.1.

In either case, we have that $a_{ij}/(b_{ij}d_i^j)$ is not (σ_y, σ_z)-summable. Since r is nonzero, at least one of the $a_{ij}/(b_{ij}d_i^j)$ is nonzero. By [35, Lemma 3.2], r is therefore not (σ_y, σ_z)-summable. $\square$

Proposition 3.7. *Let $r \in \mathbb{F}(y,z)$ be a nonzero (σ_y, σ_z)-remainder with the form (3.4), in which a_{ij} and $b_{ij}d_i^j$ are further assumed to be coprime. Assume that there exists another $r' \in \mathbb{F}(y,z)$ such that $r' - r$ is (σ_y, σ_z)-summable. Write r' in the form*

$$r' = p' + \sum_{i=1}^{m'} \sum_{j=1}^{n'_i} \frac{a'_{ij}}{b'_{ij} {d'_i}^j},$$

where $m', n'_i \in \mathbb{N}$, $p' \in \mathbb{F}(y)[z]$, $a'_{ij}, d'_i \in \mathbb{F}[y,z]$ and $b'_{ij} \in \mathbb{F}[y]$ with $\deg_z(a'_{ij}) < \deg_z(d'_i)$ and d'_i being monic irreducible factor of the denominator of r' and of positive degree in z. For each $1 \le i \le m$, define

$$\Lambda_i = \{i' \in \mathbb{N} \mid 1 \le i' \le m' \text{ and } d'_{i'} = \sigma_y^{\lambda_{i'}} \sigma_z^{\mu_{i'}}(d_i) \text{ for } \lambda_{i'}, \mu_{i'} \in \mathbb{Z}\}.$$

Then Λ_i is nonempty for any $1 \le i \le m$. Moreover, $m \le m'$, $n_i \le n'_{i'}$ for all $i' \in \Lambda_i$, $\deg_y(b_{ij}) \le \sum_{i' \in \Lambda_i} \deg_y(b'_{i'j})$ for each $1 \le i \le m$ and $1 \le j \le n_i$, and the degree in z of the denominator of r is no more than that of r'.

Proof. Since $r' - r$ is (σ_y, σ_z)-summable, all the rational function $\sum_{i' \in \Lambda_i} a'_{i'j}/(b'_{i'j}{d'_{i'}}^j) - a_{ij}/(b_{ij}d_i^j)$ are (σ_y, σ_z)-summable by [35, Lemma 3.2], and then so are the

$$\sum_{i'\in\Lambda_i} \frac{\sigma_y^{-\lambda_{i'}}\sigma_z^{-\mu_{i'}}(a'_{i'j})}{\sigma_y^{-\lambda_{i'}}(b'_{i'j})d_i^j} - \frac{a_{ij}}{b_{ij}d_i^j}. \tag{3.5}$$

Since r is a nonzero (σ_y, σ_z)-remainder, we conclude from Proposition 3.6 that each nonzero $a_{ij}/(b_{ij}d_i^j)$ is not (σ_y, σ_z)-summable. Notice that for each $1 \le i \le m$, there is at least one integer j with $1 \le j \le n_i$ such that $a_{ij} \neq 0$. It then follows from the summability of (3.5) that every Λ_i is nonempty, namely every d_i is (y, z)-shift equivalent to some $d'_{i'}$ for $1 \le i' \le m'$, and that $n_i \le n'_{i'}$ for any $i' \in \Lambda_i$. Notice that the d_i are pairwise (y, z)-shift inequivalent. Thus the Λ_i are pairwise disjoint, which implies that $m \le m'$. Accordingly, the degree in z of the denominator of r is no more than that of r'.

It remains to show the inequality for the degree of each b_{ij}. For each $1 \le i \le m$ and $1 \le j \le n_i$, by Theorem 3.1, the summability of (3.5) either yields

$$\sum_{i'\in\Lambda_i} \frac{\sigma_y^{-\lambda_{i'}}\sigma_z^{-\mu_{i'}}(a'_{i'j})}{\sigma_y^{-\lambda_{i'}}(b'_{i'j})} = \sigma_y^{\beta}\sigma_z^{-\alpha}(q) - q + \frac{a_{ij}}{b_{ij}} \quad \text{for some } q \in \mathbb{F}(y)[z],$$

if d_i is (y, z)-integer linear over $\mathbb{F}$ of (α, β)-type or otherwise yields

$$\sum_{i'\in\Lambda_i} \frac{\sigma_y^{-\lambda_{i'}}\sigma_z^{-\mu_{i'}}(a'_{i'j})}{\sigma_y^{-\lambda_{i'}}(b'_{i'j})} = \frac{a_{ij}}{b_{ij}}.$$

The assertion is evident in the latter case. For the former case, because $a_{ij}/(b_{ij}d_i^j)$ is a remainder fraction, the assertion follows by the minimality of $\phi_{\alpha,\beta}(b_{ij})$ (and thus b_{ij}) from Theorem 2.1. $\square$

With everything in place, we now present a bivariate extension of the Abramov reduction, which addresses the (σ_y, σ_z)-decomposition problem.

Bivariate Abramov reduction. Given a rational function $f \in \mathbb{F}(y, z)$, compute three rational functions $g, h, r \in \mathbb{F}(y, z)$ such that r is a (σ_y, σ_z)-remainder of f and

$$f = (\mathrm{S}_y - 1)(g) + (\mathrm{S}_z - 1)(h) + r. \tag{3.6}$$

1. Apply the primary reduction to f to find $g, h \in \mathbb{F}(y, z)$, $m, n_i \in \mathbb{N}$, $a_{ij}, d_i \in \mathbb{F}[y, z]$ and $b_{ij} \in \mathbb{F}[y]$ such that (3.1) holds.
2. For $i = 1, \ldots, m$ do

   If d_i is (y, z)-integer linear over $\mathbb{F}$ of (α_i, β_i)-type then

   2.1 Compute $\tilde{a}_{ij}/\tilde{b}_{ij} = \phi_{\alpha_i,\beta_j}(a_{ij}/b_{ij})$ with ϕ_{α_i,β_i} being the map for (α_i, β_i)-shift reduction;

   2.2 For $j = 1, \ldots, n_i$ do

   2.2.1 Apply the Abramov reduction to $\tilde{a}_{ij}/\tilde{b}_{ij}$ with respect to y to get $\tilde{q}_{ij}, \tilde{r}_{ij} \in \mathbb{F}(y)[z]$ such that

$$\frac{\tilde{a}_{ij}}{\tilde{b}_{ij}} = \sigma_y(\tilde{q}_{ij}) - \tilde{q}_{ij} + \tilde{r}_{ij}.$$

2.2.2 Apply $\phi^{-1}_{\alpha_i,\beta_i}$ to both sides of the previous equation to get

$$\frac{a_{ij}}{b_{ij}} = \sigma_y^{\beta_i}\sigma_z^{-\alpha_i}(q_{ij}) - q_{ij} + r_{ij}, \tag{3.7}$$

where $q_{ij} = \phi^{-1}_{\alpha_i,\beta_i}(\tilde{q}_{ij})$ and $r_{ij} = \phi^{-1}_{\alpha_i,\beta_i}(\tilde{r}_{ij})$.

2.2.3 Update a_{ij} and b_{ij} to be the numerator and denominator of r_{ij}, respectively.

2.3 Update

$$g = g + \sum_{j=1}^{n_i}\sum_{k=0}^{\beta_i-1}\sigma_y^k\sigma_z^{-\alpha_i}\left(\frac{q_{ij}}{d_i^j}\right) \quad\text{and}\quad h = h + \begin{cases} \displaystyle\sum_{j=1}^{n_i}\sum_{k=1}^{\alpha_i}\sigma_z^{-k}\left(-\frac{q_{ij}}{d_i^j}\right) & \alpha_i \geq 0 \\ \displaystyle\sum_{j=1}^{n_i}\sum_{k=0}^{-\alpha_i-1}\sigma_z^{k}\left(\frac{q_{ij}}{d_i^j}\right) & \alpha_i < 0 \end{cases}. \tag{3.8}$$

3. Update $r = \sum_{i=1}^{m}\sum_{j=1}^{n_i} a_{ij}/(b_{ij}d_i^j)$, and return g, h, r.

Theorem 3.8. *Let f be a rational function in $\mathbb{F}(y,z)$. Then the bivariate Abramov reduction computes two rational functions $g, h \in \mathbb{F}(y,z)$ and a (σ_y,σ_z)-remainder $r \in \mathbb{F}(y,z)$ such that (3.6) holds. Moreover, f is (σ_y,σ_z)-summable if and only if $r = 0$.*

Proof. Applying the primary reduction to f yields (3.1). For any nonzero $a_{ij}/(b_{ij}d_i^j)$ obtained in step 1, if d_i is not (y,z)-integer linear over $\mathbb{F}$ then we know from Theorem 3.1 that $a_{ij}/(b_{ij}d_i^j)$ is not (σ_y,σ_z)-summable and is thus already a remainder fraction. Otherwise, there exist coprime integers α_i, β_i with $\beta_i > 0$ so that $d_i = p_i(\alpha_i y + \beta_i z)$ for some $p_i(Z) \in \mathbb{F}[Z]$. By Theorem 2.1 and Definition 3.4, for each $1 \leq j \leq n_i$, steps 2.2.1-2.2.2 correctly find q_{ij} and r_{ij} such that (3.7) holds and r_{ij}/d_i^j is a remainder fraction. After step 2.2, plugging all (3.7) into (3.1) then gives (with a slight abuse of notation):

$$f = (\mathrm{S}_y - 1)(g) + (\mathrm{S}_z - 1)(h) + \sum_{i:\, d_i = p_i(\alpha_i y + \beta_i z)}\sum_{j=1}^{n_i}\frac{\sigma_y^{\beta_i}\sigma_z^{-\alpha_i}(q_{ij}) - q_{ij}}{d_i^j} + r,$$

where the index i runs through all (y,z)-integer linear d_i's and $r = \sum_{i=1}^{m}\sum_{j=1}^{n_i} a_{ij}/(b_{ij}d_i^j)$ is a (σ_y,σ_z)-remainder by Definition 3.5. The assertions then follow from Proposition 3.6 and the observation that

$$\frac{\sigma_y^{\beta_i}\sigma_z^{-\alpha_i}(q_{ij}) - q_{ij}}{d_i^j} = (\mathrm{S}_y - 1)\left(\sum_{k=0}^{\beta_i-1}\sigma_y^k\sigma_z^{-\alpha_i}\left(\frac{q_{ij}}{d_i^j}\right)\right)$$

$$
+ \begin{cases} (\mathrm{S}_z - 1)\left(\sum_{k=1}^{\alpha_i} \sigma_z^{-k}\left(-\frac{q_{ij}}{d_i^j}\right)\right) & \text{if } \alpha_i \geq 0 \\ (\mathrm{S}_z - 1)\left(\sum_{k=0}^{-\alpha_i-1} \sigma_z^{k}\left(\frac{q_{ij}}{d_i^j}\right)\right) & \text{if } \alpha_i < 0 \end{cases} \tag{3.9}
$$

for any $d_i = p_i(\alpha_i y + \beta_i z)$. □

Example 3.9. Consider the rational function f admitting the partial fraction decomposition $f = f_1 + f_2 + f_3$ with

$$
f_1 = \frac{x^2 z + 1}{\underbrace{(x+y)(x+z)^2 + 1}_{d_1}}, \quad f_2 = \frac{(x^2 + xy + 3x - 3)z - x - y + 3}{(x+y)(x+y+3)(\underbrace{(x+2y+3z)^2+1}_{d_2})}
$$

$$
\text{and} \quad f_3 = \frac{1}{\underbrace{x - y + z}_{d_3}}.
$$

Note that d_1, d_2, d_3 are (y, z)-shift inequivalent to each other. Hence f remains unchanged after applying the primary reduction. Since d_1 is not (y, z)-integer linear, we leave f_1 untouched and proceed to deal with f_2. Notice that d_2 is (y, z)-integer linear of $(2, 3)$-type. Then applying the Abramov reduction to $\phi_{2,3}(f_2 d_2)$ with $\phi_{2,3}$ being the map for $(2, 3)$-shift reduction yields

$$
\phi_{2,3}(f_2 d_2) = (\mathrm{S}_y - 1)\left(\frac{z - 6xy^2 - 2x^2 y + 2x}{3(x+3y)}\right) + \frac{\frac{1}{3}xz + \frac{2}{3}x^2 + 1}{x + 3y},
$$

which, when applied by $\phi_{2,3}^{-1}$, leads to

$$
f_2 d_2 = (\mathrm{S}_y^3 \mathrm{S}_z^{-2} - 1)(q_2) + \frac{\frac{1}{3}x(2y+3z) + \frac{2}{3}x^2 + 1}{x + y} \quad \text{with}
$$

$$
q_2 = \frac{3(2y+3z) - 2xy^2 - 2x^2 y + 6x}{9(x+y)}.
$$

Using (3.9), we decompose f_2 as

$$
f_2 = (\mathrm{S}_y - 1)\left(\sum_{k=0}^{2} \sigma_y^k \sigma_z^{-2}\Big(\frac{q_2}{d_2}\Big)\right) + (\mathrm{S}_z - 1)\left(\sum_{k=1}^{2} \sigma_z^{-k}\Big(-\frac{q_2}{d_2}\Big)\right)
$$

$$
+ \frac{\frac{1}{3}x(2y+3z) + \frac{2}{3}x^2 + 1}{\underbrace{(x+y)((x+2y+3z)^2+1)}_{r}}.
$$

One sees that r is a (σ_y, σ_z)-remainder of f_2, and thus f_2 is not (σ_y, σ_z)-summable by Theorem 3.8. Along the same lines as above, we have

$$f_3 = (\mathrm{S}_y - 1)\left(\frac{y}{x-y+z+1}\right) + (\mathrm{S}_z - 1)\left(\frac{y}{x-y+z}\right),$$

implying that f_3 is (σ_y, σ_z)-summable. Combining everything together, f is finally decomposed as

$$f = (\mathrm{S}_y - 1)(g) + (\mathrm{S}_z - 1)(h) + f_1 + r$$

with $g = \sum_{k=0}^{2} \sigma_y^k \sigma_z^{-2}(q_2/d_2) + y/(x-y+z+1)$ and $h = \sum_{k=1}^{2} \sigma_z^{-k}(-q_2/d_2) + y/(x-y+z)$. Thus f is not (σ_y, σ_z)-summable by Theorem 3.8. We will use f as a running example in this paper.

4. Linearity of remainders

As mentioned in the introduction, we expect our reduction algorithm to induce a linear map, that is, the sum of two remainders was expected to also be a remainder. Unfortunately, this is not always the case in our setting, because some requirements in Definition 3.5 may be broken by the addition among (σ_y, σ_z)-remainders, as seen in the following examples. This prevents us from applying the bivariate Abramov reduction developed in the previous section to construct a telescoper in a direct way as was done in the differential case. However, observe that a rational function in $\mathbb{F}(y, z)$ may have more than one (σ_y, σ_z)-remainder and any two of them differ by a (σ_y, σ_z)-summable rational function. This suggests a possible way to circumvent the above difficulty. That is, choosing proper members from the residue class modulo summable rational functions of the given (σ_y, σ_z)-remainders so as to make the linearity become true. The goal of this section is to show that this direction always works and it can be accomplished algorithmically. We note that a similar idea was used in the bivariate hypergeometric case [24, §5].

Example 4.1. Let $r = f_1$ and $s = \sigma_x(f_1)$ with f_1 being given in Example 3.9. Then r and s are both (σ_y, σ_z)-remainders since both denominators d_1 and $\sigma_x(d_1)$ are not (y, z)-integer linear. However their sum is not a (σ_y, σ_z)-remainder since d_1 is (y, z)-shift equivalent to $\sigma_x(d_1)$, namely $d_1 = \sigma_y^{-1}\sigma_z^{-1}\sigma_x(d_1)$. Nevertheless, we can find another (σ_y, σ_z)-remainder t of s such that $r + t$ has this property. For example, let

$$t = (\mathrm{S}_y - 1)\left(-\sigma_y^{-1}(s)\right) + (\mathrm{S}_z - 1)\left(-\sigma_y^{-1}\sigma_z^{-1}(s)\right) + s = \frac{(x+1)^2(z-1)+1}{(x+y)(x+z)^2+1},$$

and then

$$r+t=\frac{(2x^2+2x+1)z-x^2-2x+1}{(x+y)(x+z)^2+1}$$

is a (σ_y,σ_z)-remainder by definition. Alternatively, one can compute a (σ_y,σ_z)-remainder $\tilde{t}$ of r, say

$$\tilde{t}=(\mathrm{S}_y-1)\,(r)+(\mathrm{S}_z-1)\,(\sigma_y(r))+r=\frac{x^2(z+1)+1}{(x+y+1)(x+z+1)^2+1}$$

so that

$$\tilde{t}+s=\frac{(2x^2+2x+1)z+x^2+2}{(x+y+1)(x+z+1)^2+1}$$

is a (σ_y,σ_z)-remainder.

Example 4.2. Let

$$r=\frac{\frac{1}{3}x(2y+3z)+\frac{2}{3}x^2+1}{(x+y)((x+2y+3z)^2+1)}\quad\text{and}\quad s=\frac{(\frac{1}{3}x+1)(2y+3z)+\frac{2}{3}(x+1)^2+2x+\frac{13}{3}}{(x+y+5)((x+2y+3z+1)^2+1)}.$$

Then both r and s are (σ_y,σ_z)-remainders, but again their sum is not since $(x+2y+3z)^2+1$ is (y,z)-shift equivalent to $(x+2y+3z+1)^2+1$. As in Example 4.1, we find a (σ_y,σ_z)-remainder

$$\tilde{s}=\frac{a/b}{(x+2y+3z)^2+1}\quad\text{with}\quad\frac{a}{b}=\frac{(\frac{1}{3}x+1)(2y+3z)+\frac{2}{3}x^2+3x+4}{x+y+6}$$

such that $s-\tilde{s}$ is (σ_y,σ_z)-summable. However, the sum $r+\tilde{s}$ is still not a (σ_y,σ_z)-remainder since $\phi_{2,3}\big((\frac{1}{3}x(2y+3z)+\frac{2}{3}x^2+1)/(x+y)+a/b\big)$ is not a σ_y-remainder, where $\phi_{2,3}$ denotes the map for $(2,3)$-shift reduction. Notice that

$$\frac{a}{b}=(\mathrm{S}_y^3\mathrm{S}_z^{-2}-1)\left(\sum_{k=1}^{2}\sigma_y^{-3k}\sigma_z^{2k}\left(\frac{a}{b}\right)\right)+\frac{(\frac{1}{3}x+1)(2y+3z)+\frac{2}{3}x^2+3x+4}{x+y},$$

so (3.9) enables us to find a new (σ_y,σ_z)-remainder

$$t=\frac{(\frac{1}{3}x+1)(2y+3z)+\frac{2}{3}x^2+3x+4}{(x+y)((x+2y+3z)^2+1)}$$

such that $s-t$ is (σ_y,σ_z)-summable and

$$r+t=\frac{(\frac{2}{3}x+1)(2y+3z)+\frac{4}{3}x^2+3x+5}{(x+y)((x+2y+3z)^2+1)}$$

is a (σ_y,σ_z)-remainder. Another possible choice is to find a (σ_y,σ_z)-remainder $\tilde{r}$ of r such that $\tilde{r}+s$ is a (σ_y,σ_z)-remainder.

In order to achieve the linearity of (σ_y, σ_z)-remainders, we need to develop two lemmas. The first one mimics the idea of Lemma 5.5 in [24] in the bivariate setting.

Lemma 4.3. *Let $a, d \in \mathbb{F}[y, z], b \in \mathbb{F}[y] \setminus \{0\}$ and $j \in \mathbb{N} \setminus \{0\}$. Let λ, μ be two integers. Then one finds $g, h \in \mathbb{F}(y, z)$ such that*

$$\frac{a}{b\, d^j} = (S_y - 1)(g) + (S_z - 1)(h) + \frac{\sigma_y^\lambda \sigma_z^\mu(a)}{\sigma_y^\lambda(b)\sigma_y^\lambda\sigma_z^\mu(d)^j}. \tag{4.1}$$

Moreover, assume that d is not (y, z)-integer linear over $\mathbb{F}$. If $a/(b\, d^j)$ is a remainder fraction, then so is $\sigma_y^\lambda\sigma_z^\mu(a)/(\sigma_y^\lambda(b)\sigma_y^\lambda\sigma_z^\mu(d)^j)$.

Proof. A direct calculation shows that

$$\frac{s}{t} = (\mathrm{S}_v - 1)\left(-\sum_{j=0}^{i-1}\sigma_v^j\left(\frac{s}{t}\right)\right) + \frac{\sigma_v^i(s)}{\sigma_v^i(t)} = (\mathrm{S}_v - 1)\left(\sum_{j=1}^{i}\sigma_v^{-j}\left(\frac{s}{t}\right)\right) + \frac{\sigma_v^{-i}(s)}{\sigma_v^{-i}(t)}$$

for any $s, t \in \mathbb{F}[y, z]$, $i \in \mathbb{N}$ and $v \in \{y, z\}$. By iteratively applying the above formulas, one readily computes $g, h \in \mathbb{F}(y, z)$ such that (4.1) holds.

Moreover, if d is not (y, z)-integer linear over $\mathbb{F}$, then neither is $\sigma_y^\lambda\sigma_z^\mu(d)$. Since $a/(b\, d^j)$ is a remainder fraction, by Definition 3.4, $\deg_z(a) < \deg_z(d)$ and d is monic, irreducible and of positive degree in z. Shifting polynomials in $\mathbb{F}[y, z]$ with respect to y or z preserves these properties. It follows from definition that $\sigma_y^\lambda\sigma_z^\mu(a)/(\sigma_y^\lambda(b)\sigma_y^\lambda\sigma_z^\mu(d)^j)$ is a remainder fraction. □

The next lemma is an immediate result of Theorem 5.6 in [24].

Lemma 4.4. *Let $\alpha, \beta \in \mathbb{Z}$ with $\beta \neq 0$ and let $\phi_{\alpha,\beta}$ denote the map for (α, β)-shift reduction. Let $a, \bar{a} \in \mathbb{F}[y, z]$ and $b, \bar{b} \in \mathbb{F}[y] \setminus \{0\}$ be such that both $\phi_{\alpha,\beta}(a/b)$ and $\phi_{\alpha,\beta}(\bar{a}/\bar{b})$ are σ_y-remainders. Then one finds $q \in \mathbb{F}(y)[z]$, $a' \in \mathbb{F}[y, z]$ and $b' \in \mathbb{F}[y]$ with $\phi_{\alpha,\beta}(a'/b')$ being a σ_y-remainder such that*

$$\frac{a}{b} = (S_y^\beta S_z^{-\alpha} - 1)(q) + \frac{a'}{b'},$$

and $\phi_{\alpha,\beta}(c_1\bar{a}/\bar{b} + c_2 a'/b')$ is a σ_y-remainder for all $c_1, c_2 \in \mathbb{F}$.

Proof. By [24, Theorem 5.6] and [36, Proposition 3.2], there exist $\tilde{q} \in \mathbb{F}(y)[z]$, $\tilde{a} \in \mathbb{F}[y, z]$ and $\tilde{b} \in \mathbb{F}[y]$ with $\tilde{a}/\tilde{b}$ being a σ_y-remainder such that

$$\phi_{\alpha,\beta}\left(\frac{a}{b}\right) = \sigma_y(\tilde{q}) - \tilde{q} + \frac{\tilde{a}}{\tilde{b}},$$

and $c_1\phi_{\alpha,\beta}(\bar{a}/\bar{b}) + c_2\tilde{a}/\tilde{b}$ is a σ_y-remainder for all $c_1, c_2 \in \mathbb{F}$. Notice that $\phi_{\alpha,\beta}$ is an $\mathbb{F}$-isomorphism and $\sigma_y \circ \phi_{\alpha,\beta} = \phi_{\alpha,\beta} \circ \tau$ with $\tau = \sigma_y^{\beta}\sigma_z^{-\alpha}$. So $\phi_{\alpha,\beta}^{-1} \circ \sigma_y = \tau \circ \phi_{\alpha,\beta}^{-1}$. Letting $q = \phi_{\alpha,\beta}^{-1}(\tilde{q})$, $a' = \phi_{\alpha,\beta}^{-1}(\tilde{a})$ and $b' = \phi_{\alpha,\beta}^{-1}(\tilde{b})$ concludes the lemma. □

We are now ready to give an algorithm that provides a feasible way to obtain the linearity.

Remainder linearization. Given two (σ_y, σ_z)-remainders $r, s \in \mathbb{F}(y, z)$, compute $g, h \in \mathbb{F}(y, z)$ and a (σ_y, σ_z)-remainder $t \in \mathbb{F}(y, z)$ such that

$$s = (\mathrm{S}_y - 1)(g) + (\mathrm{S}_z - 1)(h) + t \tag{4.2}$$

and $c_1 r + c_2 t$ is a (σ_y, σ_z)-remainder for all $c_1, c_2 \in \mathbb{F}$.

1. Write r and s in the remainder forms

$$r = \sum_{i=1}^{\bar{m}} \sum_{j=1}^{\bar{n}_i} \frac{\bar{a}_{ij}}{\bar{b}_{ij}\bar{d}_i^j} \quad \text{and} \quad s = \sum_{i=1}^{m} \sum_{j=1}^{n_i} \frac{a_{ij}}{b_{ij}d_i^j}. \tag{4.3}$$

2. Set $g = h = 0$.
   For $i = 1, \ldots, m$ do
   If there exists $k \in \{1, 2, \ldots, \bar{m}\}$ such that $\bar{d}_k = \sigma_y^{\lambda}\sigma_z^{\mu}(d_i)$ for some $\lambda, \mu \in \mathbb{Z}$, then
   2.1 For $j = 1, \ldots, n_i$ do
   2.1.1 Apply Lemma 4.3 to $a_{ij}/(b_{ij}d_i^j)$ to find $g_{ij}, h_{ij} \in \mathbb{F}(y, z)$ such that

$$\frac{a_{ij}}{b_{ij}d_i^j} = (\mathrm{S}_y - 1)(g_{ij}) + (\mathrm{S}_z - 1)(h_{ij}) + \frac{\sigma_y^{\lambda}\sigma_z^{\mu}(a_{ij})}{\sigma_y^{\lambda}(b_{ij})\bar{d}_k^j}. \tag{4.4}$$

   2.1.2 If d_i is (y, z)-integer linear over $\mathbb{F}$ of (α_i, β_i)-type then
   Apply Lemma 4.4 to $\sigma_y^{\lambda}\sigma_z^{\mu}(a_{ij})/\sigma_y^{\lambda}(b_{ij})$ to find $q_{ij} \in \mathbb{F}(y)[z]$, $a'_{ij} \in \mathbb{F}[y, z]$ and $b'_{ij} \in \mathbb{F}[y]$ with $\phi_{\alpha_i,\beta_i}(a'_{ij}/b'_{ij})$ being a σ_y-remainder such that

$$\frac{\sigma_y^{\lambda}\sigma_z^{\mu}(a_{ij})}{\sigma_y^{\lambda}(b_{ij})} = (\mathrm{S}_y^{\beta_i}\mathrm{S}_z^{-\alpha_i} - 1)(q_{ij}) + \frac{a'_{ij}}{b'_{ij}}, \tag{4.5}$$

   and $\phi_{\alpha_i,\beta_i}(c_1\bar{a}_{kj}/\bar{b}_{kj} + c_2 a'_{ij}/b'_{ij})$ is a σ_y-remainder for all $c_1, c_2 \in \mathbb{F}$; update a_{ij}, b_{ij} to be a'_{ij}, b'_{ij}, respectively, and update g, h by (3.8).
   Else update a_{ij}, b_{ij} to be $\sigma_y^{\lambda}\sigma_z^{\mu}(a_{ij}), \sigma_y^{\lambda}(b_{ij})$, respectively.
   2.2 Update d_i to be $\bar{d}_k$, and update $g = g + \sum_{j=1}^{n_i} g_{ij}$, $h = h + \sum_{j=1}^{n_i} h_{ij}$.
3. Set $t = \sum_{i=1}^{m}\sum_{j=1}^{n_i} a_{ij}/(b_{ij}d_i^j)$, and return g, h, t.

Theorem 4.5. *Let r and s be two (σ_y, σ_z)-remainders. Then the remainder linearization correctly finds two rational functions $g, h \in \mathbb{F}(y, z)$ and a (σ_y, σ_z)-remainder $t \in \mathbb{F}(y, z)$ such that (4.2) holds and $c_1 r + c_2 t$ is a (σ_y, σ_z)-remainder for all $c_1, c_2 \in \mathbb{F}$.*

Proof. Since both r and s are (σ_y, σ_z)-remainders, they admit the remainder forms (4.3). For any d_i from s, if there exists some $\bar{d}_k$ from r such that $\bar{d}_k = \sigma_y^\lambda \sigma_z^\mu (d_i)$ for some $\lambda, \mu \in \mathbb{Z}$, then for each integer j with $1 \leq j \leq n_i$, one sees from Lemma 4.3 that step 2.1.1 correctly finds the g_{ij}, h_{ij} such that (4.4) holds. Moreover, $\sigma_y^\lambda \sigma_z^\mu (a_{ij})/(\sigma_y^\lambda (b_{ij}) \bar{d}_k^j)$ is a remainder fraction if d_i is not (y, z)-integer linear over $\mathbb{F}$. When d_i is (y, z)-integer linear over $\mathbb{F}$ of (α_i, β_i)-type, Lemma 4.4 assures that (4.5) holds and $a'_{ij}/(b'_{ij} \bar{d}_k^j)$ is a remainder fraction. Note that $d_1, \ldots, d_m$ are pairwise (y, z)-shift inequivalent since s is a (σ_y, σ_z)-remainder. Also note that each d_i can only be replaced by some $\bar{d}_k$ which is (y, z)-shift equivalent to d_i every time the algorithm passes through step 2.2. Thus the updated d_i after step 2 remain to be (y, z)-shift inequivalent to each other. It then follows from Definition 3.5 that $t = \sum_{i=1}^m \sum_{j=1}^{n_i} a_{ij}/(b_{ij} {d_i}^j)$ in step 3 (with a slight abuse of notations) is a (σ_y, σ_z)-remainder. Substituting all equations (4.4)-(4.5) into (4.3), together with (3.9), immediately yields (4.2).

Let $c_1, c_2 \in \mathbb{F}$. Then it remains to prove that $c_1 r + c_2 t$ is a (σ_y, σ_z)-remainder. Notice that for any two remainder fractions: $\bar{a}_{kj}/(\bar{b}_{kj} \bar{d}_k^j)$ from r and $a_{ij}/(b_{ij} d_i^j)$ from t with $\bar{d}_k \nsim_{y,z} d_i$, it is readily seen from definition that their any linear combination over $\mathbb{F}$ is again a remainder fraction. Thus it amounts to showing that $c_1 \bar{a}_{kj}/(\bar{b}_{kj} \bar{d}_k^j) + c_2 a_{ij}/(b_{ij} d_i^j)$ is a remainder fraction in the case when $\bar{d}_k \sim_{y,z} d_i$. We know from step 2 that in this case $d_i = \bar{d}_k$, and $\phi_{\alpha_i, \beta_i}(c_1 \bar{a}_{kj}/\bar{b}_{kj} + c_2 a_{ij}/b_{ij})$ is a σ_y-remainder if d_i is (y, z)-integer linear over $\mathbb{F}$ of (α_i, β_i)-type. Therefore, the theorem is concluded by definition. $\square$

5. Telescoping via reduction

Recall that a telescoper L, for a given rational function $f \in \mathbb{F}(y, z)$, is a nonzero operator in $\mathbb{F}[\mathrm{S}_x]$ such that $L(f)$ is (σ_y, σ_z)-summable. For discrete trivariate rational functions, telescopers do not always exist. Recently, a criterion for determining the existence of telescopers in this case was presented in the work [23]. In order to adapt it into our setting, we will consider primitive parts of polynomials in $\mathbb{F}[y]$. Let $p \in \mathbb{F}[y]$ be of the form $p = \sum_{i=0}^d (a_i/b) y^i$ for $d \in \mathbb{N}$ and $a_i, b \in \mathbb{K}[x]$ with $b \neq 0$. Then the *content* $\operatorname{cont}_y(p)$ of p with respect to y is defined as $\operatorname{cont}_y(p) = \gcd(a_0, \ldots, a_d)/b \in \mathbb{F}$, and the corresponding *primitive part* $\operatorname{prim}_y(p) = p/\operatorname{cont}_y(p)$. For example, by letting $p = 3xy - 9x/(x + 1) \in \mathbb{F}[y]$, we have $\operatorname{cont}_y(p) = 3x/(x + 1) \in \mathbb{F}$ and $\operatorname{prim}_y(p) = (x + 1)y - 3 \in \mathbb{K}[x, y]$. Evidently, $\operatorname{prim}_y(p)$ is a polynomial in $\mathbb{K}[x, y]$ whose coefficients with respect to y have no nonconstant common divisors in $\mathbb{K}[x]$.

We summarize the existence criterion for telescopers from [23] in the following

Theorem 5.1 *(Existence criterion). Let f be a rational function in $\mathbb{F}(y, z)$. Assume that applying the bivariate Abramov reduction to f yields (3.6), where $g, h, r \in \mathbb{F}(y, z)$ and r*

is a (σ_y, σ_z)*-remainder with the remainder form* (3.4)*. Then* f *has a telescoper if and only if for each* $1 \le i \le m$ *and* $1 \le j \le n_i$*,*

(i) *there exists a positive integer* ξ_i *such that* $\sigma_x^{\xi_i}(d_i) = \sigma_y^{\zeta_i}\sigma_z^{\eta_i}(d_i)$ *for some integers* ζ_i, η_i*,*
(ii) *and* b_{ij} *is* (x, y)*-integer linear over* $\mathbb{K}$*, in particular,* $\sigma_x^{\xi_i}(\mathrm{prim}_y(b_{ij}))=\sigma_y^{\zeta_i}(\mathrm{prim}_y(b_{ij}))$ *if* d_i *is not* (y, z)*-integer linear over* $\mathbb{F}$*.*

Algorithms for checking the conditions (i)-(ii) were also described in the same paper [23]. With termination guaranteed by the above criterion, we now use the bivariate Abramov reduction to develop a creative telescoping algorithm in the spirit of the general reduction-based approach.

Algorithm ReductionCT. Given a rational function $f \in \mathbb{F}(y, z)$, compute a minimal telescoper $L \in \mathbb{F}[\mathrm{S}_x]$ for f and a corresponding certificate $(g, h) \in \mathbb{F}(y, z)^2$ when telescopers exist.

1. Apply the bivariate Abramov reduction to f to find $g_0, h_0 \in \mathbb{F}(y, z)$ and a (σ_y, σ_z)-remainder $r_0 \in \mathbb{F}(y, z)$ such that

$$f = (\mathrm{S}_y - 1)(g_0) + (\mathrm{S}_z - 1)(h_0) + r_0. \tag{5.1}$$

2. If $r_0 = 0$ then set $L = 1$, $(g, h) = (g_0, h_0)$ and return.
3. If conditions (i)-(ii) in Theorem 5.1 are not satisfied, then return “No telescopers exist”.
4. Set $R = u_0 r_0$, where u_0 is an indeterminate.
   For $\ell = 1, 2, \ldots$ do
   4.1 Apply the remainder linearization to $\sigma_x(r_{\ell-1})$ with respect to R to find $g_\ell, h_\ell \in \mathbb{F}(y, z)$ and a (σ_y, σ_z)-remainder $r_\ell \in \mathbb{F}(y, z)$ such that

$$\sigma_x(r_{\ell-1}) = (\mathrm{S}_y - 1)(g_\ell) + (\mathrm{S}_z - 1)(h_\ell) + r_\ell, \tag{5.2}$$

   and that $R + u_\ell r_\ell$ is a (σ_y, σ_z)-remainder, where u_ℓ is an indeterminate.
   4.2 Update $R = R + u_\ell r_\ell$ and update $g_\ell = g_\ell + \sigma_x(g_{\ell-1})$, $h_\ell = h_\ell + \sigma_x(h_\ell)$ so that

$$\sigma_x^\ell(f) = (\mathrm{S}_y - 1)(g_\ell) + (\mathrm{S}_z - 1)(h_\ell) + r_\ell. \tag{5.3}$$

   4.3 If there exist nontrivial $c_0, \ldots, c_\ell \in \mathbb{F}$ such that $R\ |_{u_i=c_i}= 0$, then set $L = \sum_{i=0}^{\ell} c_i \mathrm{S}_x^i$ and $(g, h) = (\sum_{i=0}^{\ell} c_i g_i, \sum_{i=0}^{\ell} c_i h_i)$, and return.

Theorem 5.2. *Let* f *be a rational function in* $\mathbb{F}(y, z)$*. Then the algorithm* ***ReductionCT*** *terminates and returns a minimal telescoper for* f *when such a telescoper exists.*

Proof. By Theorems 3.8 and 5.1, steps 2-3 are correct. Because r_0 is a (σ_y, σ_z)-remainder, so is its shift $\sigma_x(r_0)$. By Theorem 4.5, step 4.1 correctly finds $g_1, h_1 \in \mathbb{F}(y, z)$ and a (σ_y, σ_z)-remainder $r_1 \in \mathbb{F}(y, z)$ such that (5.2) holds for $\ell = 1$ and $R + u_1 r_1 = u_0 r_0 + u_1 r_1$ is a (σ_y, σ_z)-remainder for all $u_0, u_1 \in \mathbb{F}$. Applying σ_x to both sides of (5.1), together with step 4.1, one sees that step 4.2 gives (5.3) for $\ell = 1$. The correctness of step 4.2 for each iteration of the loop in step 4 then follows by induction on ℓ.

If f does not have a telescoper then the algorithm halts after step 3. Otherwise, telescopers for f exist by Theorem 5.1. Let $L = \sum_{\ell=0}^{\rho} c_\ell \mathrm{S}_x^\ell \in \mathbb{F}[\mathrm{S}_x]$ be a telescoper for f of minimal order. Then $c_\rho \neq 0$ and by (5.3), applying L to f gives

$$L(f) = \sum_{\ell=0}^{\rho} c_\ell \sigma_x^\ell(f) = (\mathrm{S}_y - 1)\left(\sum_{\ell=0}^{\rho} c_\ell g_\ell\right) + (\mathrm{S}_z - 1)\left(\sum_{\ell=0}^{\rho} c_\ell h_\ell\right) + \sum_{\ell=0}^{\rho} c_\ell r_\ell.$$

Notice that $\sum_{\ell=0}^{\rho} c_\ell r_\ell$ is a (σ_y, σ_z)-remainder by step 4.1. It follows from Theorem 3.8 that $L(f)$ is (σ_y, σ_z)-summable, namely L is a telescoper for f, if and only if $\sum_{\ell=0}^{\rho} c_\ell r_\ell = 0$. This implies that the linear system over $\mathbb{F}$ with unknowns u_ℓ obtained by equating $\sum_{\ell=0}^{\rho} u_\ell r_\ell$ to zero has a nontrivial solution, which yields a telescoper of minimal order. The algorithm thus terminates. □

In what follows, we describe an alternative way, in addition to the above algorithm, for creative telescoping in our trivariate rational setting. As such, we need the notion of least common left multiples. Recall that an operator $L \in \mathbb{F}[\mathrm{S}_x]$ is a *common left multiple* of operators $L_1, \ldots, L_m \in \mathbb{F}[\mathrm{S}_x]$ if there exist operators $L_1', \ldots, L_m' \in \mathbb{F}[\mathrm{S}_x]$ such that $L = L_1' L_1 = \cdots = L_m' L_m$. Amongst all such common left multiples, the monic one of lowest possible degree with respect to S_x is called the *least common left multiple*. Many efficient algorithms for computing the least common left multiple of operators are available; see [8] and the references therein.

The following lemma is an immediate extension of [40, Theorem 2] to the trivariate case, and thus we omit the proof.

Lemma 5.3. *Let $r = r_1 + \cdots + r_m$ with $r_i \in \mathbb{F}(y, z)$ and let $L_1, \ldots, L_m \in \mathbb{F}[\mathrm{S}_x]$ be the respective minimal telescopers for $r_1, \ldots, r_m$. Then the least common left multiple L of the L_i is a telescoper for r. Moreover, if any telescoper for r is also a telescoper for each r_i with $1 \leq i \leq m$, then L is a minimal telescoper for r.*

The following proposition shows that the least common multiple gives a minimal telescoper for the given sum provided that the denominators of distinct summands are comprised of distinct (x, y, z)-shift equivalence classes.

Proposition 5.4. *Let $r \in \mathbb{F}(y, z)$ be a rational function of the form*

$$r = r_1 + r_2 + \cdots + r_m,$$

where $r_i = a_i/d_i$ with $a_i, d_i \in \mathbb{F}[y,z]$ satisfying the conditions

(i) $\deg_z(a_i) < \deg_z(d_i)$;
(ii) *any monic irreducible factor of d_i of positive degree in z is (x,y,z)-shift inequivalent to all factors of $d_{i'}$ whenever $1 \le i, i' \le m$ and $i \ne i'$.*

Let $L_1, \dots, L_m \in \mathbb{F}[\mathrm{S}_x]$ be the respective minimal telescopers for $r_1, \dots, r_m$. Then the least common left multiple L of the L_i is a minimal telescoper for r. Moreover, for each $1 \le i \le m$, let (g_i, h_i) be a corresponding certificate for L_i and let $L_i' \in \mathbb{F}[\mathrm{S}_x]$ be the cofactor of L_i so that $L = L_i' L_i$. Then

$$\left(\sum_{i=1}^{m} L_i'(g_i), \sum_{i=1}^{m} L_i'(h_i) \right)$$

is a corresponding certificate for L.

Proof. Let $\tilde{L} \in \mathbb{F}[\mathrm{S}_x]$ be a telescoper for r. In order to show the first assertion, by Lemma 5.3, it suffices to verify that $\tilde{L}$ is also a telescoper for each r_i with $1 \le i \le m$. Notice that the application of a nonzero operator from $\mathbb{F}[\mathrm{S}_x]$ does not change the (x,y,z)-shift equivalence classes, with representatives being monic irreducible polynomials of positive degrees in z, that appear in a given polynomial in $\mathbb{F}[y,z]$. Hence condition (ii) remains valid when d_i and $d_{i'}$ are replaced by $\tilde{L}(d_i)$ and $\tilde{L}(d_{i'})$, respectively. It then follows that any two monic irreducible factors of positive degrees in z from distinct d_i are (y,z)-shift inequivalent to each other. By the definition of telescopers, $\tilde{L}(r)$ is (σ_y, σ_z)-summable, and then so is each $\tilde{L}(r_i)$ according to [35, Lemma 3.2]. This implies that $\tilde{L}$ is indeed a telescoper for each r_i. The second assertion follows by observing that $(\mathrm{S}_y - 1)$ and $(\mathrm{S}_z - 1)$ both commute with operators from $\mathbb{F}[\mathrm{S}_x]$. $\square$

The above proposition suggests a natural approach to construct a minimal telescoper for a given rational function. More precisely, let $f \in \mathbb{F}(y,z)$ and assume that applying the bivariate Abramov reduction to f yields (3.6) with r admitting the remainder form (3.4). The approach proceeds by separately taking each $\sum_{j=1}^{n_i} a_{ij}/(b_{ij} d_i^j)$ in (3.4) as the basic case and computes its own minimal telescoper $L_i \in \mathbb{F}[\mathrm{S}_x]$ using the algorithm **ReductionCT**, and then returns the least common left multiple L of all L_i as the output. By Proposition 5.4, such an L gives a minimal telescoper for r (and thus for f). We refer to this approach as the LCLM version of our algorithm **ReductionCT**.

5.1. Examples

Example 5.5. Consider the rational function f_1 given in Example 3.9. Note that f_1 is a remainder fraction and satisfies conditions (i)-(ii) in Theorem 5.1. So telescopers for f_1 exist. Applying the algorithm **ReductionCT** to f_1, we obtain in step 4 that

$$\sigma_x^\ell(f_1) = (\mathrm{S}_y - 1)(g_\ell) + (\mathrm{S}_z - 1)(h_\ell) + r_\ell \quad \text{for } \ell = 0, 1, 2,$$

where

$$r_0 = f_1, \quad r_1 = \frac{(x+1)^2(z-1)+1}{(x+y)(x+z)^2+1}, \quad r_2 = \frac{(x+2)^2(z-2)+1}{(x+y)(x+z)^2+1}$$

and $g_\ell, h_\ell \in \mathbb{F}(y,z)$ are not displayed here to keep things neat. By finding an $\mathbb{F}$-linear dependency among r_0, r_1, r_2, we see that

$$\begin{aligned} L_1 &= (x^4+2x^3+x^2+2x+1)S_x^2 - 2(x^4+4x^3+4x^2+2x+2)S_x \\ &\quad + (x^4+6x^3+13x^2+14x+7) \end{aligned}$$

is a minimal telescoper for f_1.

Example 5.6. Consider the rational function f_2 given in Example 3.9, which can be decomposed as

$$f_2 = (\mathrm{S}_y - 1)(g_0) + (\mathrm{S}_z - 1)(h_0) + \underbrace{\frac{\frac{1}{3}x(2y+3z)+\frac{2}{3}x^2+1}{(x+y)((x+2y+3z)^2+1)}}_{r_0} \quad \text{for some } g_0, h_0 \in \mathbb{F}(y,z).$$

Note that r_0 is a remainder fraction and satisfies conditions (i)-(ii) in Theorem 5.1. Thus telescopers for f_2 exist. Applying the algorithm **ReductionCT** to f_2, we obtain in step 4 that

$$\sigma_x^\ell(f_2) = (\mathrm{S}_y - 1)(g_\ell) + (\mathrm{S}_z - 1)(h_\ell) + r_\ell \quad \text{for } \ell = 0, 1, \ldots, 6,$$

where $g_\ell, h_\ell \in \mathbb{F}(y,z)$ are again not displayed due to the large sizes, and

$$\begin{aligned} r_1 &= \tfrac{(\frac{1}{3}x+1)(2y+3z)+\frac{2}{3}x^2+x+\frac{4}{3}}{(x+y+2)((x+2y+3z)^2+1)}, & r_2 &= \tfrac{(\frac{1}{3}x+\frac{2}{3})(2y+3z)+\frac{2}{3}x^2+2x+\frac{7}{3}}{(x+y+4)((x+2y+3z)^2+1)}, \\ r_3 &= \tfrac{(\frac{1}{3}x+1)(2y+3z)+\frac{2}{3}x^2+4}{(x+y)((x+2y+3z)^2+1)}, & r_4 &= \tfrac{(\frac{1}{3}x+\frac{4}{3})(2y+3z)+\frac{2}{3}x^2+4x+\frac{19}{3}}{(x+y+2)((x+2y+3z)^2+1)}, \\ r_5 &= \tfrac{(\frac{1}{3}x+\frac{5}{3})(2y+3z)+\frac{2}{3}x^2+5x+\frac{28}{3}}{(x+y+4)((x+2y+3z)^2+1)}, & r_6 &= \tfrac{(\frac{1}{3}x+2)(2y+3z)+\frac{2}{3}x^2+6x+13}{(x+y)((x+2y+3z)^2+1)}. \end{aligned}$$

Then one finds an $\mathbb{F}$-linear dependency among r_0, r_3, r_6 which yields a minimal telescoper

$$L_2 = (x^2+3x-3)S_x^6 - 2(x^2+6x-3)S_x^3 + x^2 + 9x + 15.$$

The following illustrates the result of Proposition 5.4.

Example 5.7. Consider the same rational function f as in Example 3.9. Then we know that f_3 is (σ_y, σ_z)-summable. Thus $L_3 = 1$ is a minimal telescoper for f_3. Let $L_1, L_2 \in$

$\mathbb{F}[\mathrm{S}_x]$ be the operators computed in Examples 5.5-5.6. It then follows that the least common left multiple L of $\{L_1, L_2, L_3\}$, given by

$$\begin{aligned} L = S_x^8 &- \tfrac{2(x^2+5x+1)(3x^2+24x+31)}{(x^2+7x+7)(3x^2+21x+19)} S_x^7 + \tfrac{(x^2+3x-3)(3x^2+27x+43)}{(x^2+7x+7)(3x^2+21x+19)} S_x^6 - \tfrac{2(x^2+10x+13)}{x^2+7x+7} S_x^5 \\ &+ \tfrac{4(3x^2+24x+31)(x^2+8x+4)}{(x^2+7x+7)(3x^2+21x+19)} S_x^4 - \tfrac{2(x^2+6x-3)(3x^2+27x+43)}{(x^2+7x+7)(3x^2+21x+19)} S_x^3 + \tfrac{x^2+13x+37}{x^2+7x+7} S_x^2 \\ &- \tfrac{2(x^2+11x+25)(3x^2+24x+31)}{(x^2+7x+7)(3x^2+21x+19)} S_x + \tfrac{(x^2+9x+15)(3x^2+27x+43)}{(x^2+7x+7)(3x^2+21x+19)}, \end{aligned}$$

is a telescoper for f. On the other hand, by directly applying the algorithm **ReductionCT** to f, one sees that L is in fact a minimal telescoper for f.

5.2. Efficiency considerations

The efficiency of Algorithm **ReductionCT** can be enhanced by incorporating two modifications in the algorithm.

Simplification of step 4.1

For each iteration of the loop in step 4, rather than using the overall (σ_y, σ_z)-remainder $R = \sum_{k=0}^{\ell-1} u_k r_k$ in step 4.1, we can apply the remainder linearization to the shift value $\sigma_x(r_{\ell-1})$ with respect to the initial (σ_y, σ_z)-remainder r_0 only. This is sufficient as, for any (σ_y, σ_z)-remainder r_ℓ of $\sigma_x(r_{\ell-1})$ with $\ell \geq 1$, if $r_0 + r_\ell$ is a (σ_y, σ_z)-remainder then so is $R + u_\ell r_\ell$, provided that the algorithm proceeds in the described iterative fashion.

The intuition for this simplification is as follows. Notice that if the algorithm continues after passing through step 3 then $r_0 \neq 0$. Since distinct (y, z)-shift equivalence classes can be tackled separately, we restrict ourselves to the case where the denominator of r_0 is of the form

$$d\, \sigma_x^{i_1}(d) \cdots \sigma_x^{i_m}(d)$$

with $d \in \mathbb{F}[y, z]$ being monic, irreducible and of positive degree in z, $i_1, \ldots, i_m$ being distinct positive integers such that $d, \sigma_x^{i_1}(d), \ldots, \sigma_x^{i_m}(d)$ are (y, z)-shift inequivalent to each other. For simplicity, we call $(0, i_1, \ldots, i_m)$ the x-shift exponent sequence of d in r_0. By Theorem 5.1, there exists a positive integer ξ such that $\sigma_x^{\xi}(d) \sim_{y,z} d$ and so we let ξ be the smallest one with such a property. Then there are only ξ many (y, z)-shift equivalence classes produced by shifting d with respect to x, with $d, \sigma_x(d), \ldots, \sigma_x^{\xi-1}(d)$ as respective representatives. Without loss of generality, we further assume that $0 < i_1 < \cdots < i_m < \xi$. For $\ell \geq 1$, let r_ℓ be the output of the remainder linearization when applied to $\sigma_x(r_{\ell-1})$ with respect to r_0. By induction on ℓ, one sees that the x-shift exponent sequence of d in r_ℓ is given by

$$(\ell, i_1 + \ell, \ldots, i_m + \ell) \mod \xi,$$

whose entries form an $(m+1)$-subset of $\{0, 1, \ldots, \xi-1\}$. It thus follows from Definition 3.5 that $R + u_\ell r_\ell$ is also a (σ_y, σ_z)-remainder.

Simplification of step 4.3

Our second modification is in step 4.3, where we first derive from $R = 0$ the individual equation for each remainder fraction $a/(b\, d^j)$ appearing in the remainder form of R, and then build a linear system over $\mathbb{F}$ from the coefficients of the numerator of the equation with respect to y and $Z = \alpha y + \beta z$, instead of y and z, in the case where d is (y, z)-integer linear of (α, β)-type. Notice that $R = u_0 r_0 + u_1 r_1 + \cdots + u_\ell r_\ell$ at the stage of step 4.3. Let $d_1, \ldots, d_m$ be all monic irreducible polynomials of positive degrees in z that appear in the denominator of R, with multiplicities $n_1, \ldots, n_m$, respectively. For $1 \le i \le m$, $1 \le j \le n_i$ and $0 \le k \le \ell$, let $a_{ij}^{(k)} \in \mathbb{F}[y, z]$ and $b_{ij}^{(k)} \in \mathbb{F}[y]$ be such that $a_{ij}^{(k)}/(b_{ij}^{(k)} d_i^j)$ is a remainder fraction appearing in the remainder form of r_k. By coprimeness among the d_i, one gets that

$$R = 0 \quad \Longleftrightarrow \quad \sum_{k=0}^{\ell} u_k \cdot \frac{a_{ij}^{(k)}}{b_{ij}^{(k)}} = 0 \quad \text{for all } i = 1, \ldots, m \text{ and } j = 1, \ldots, n_i.$$

If d_i is (y, z)-integer linear of (α_i, β_i)-type, then $\phi_{\alpha_i,\beta_i}(a_{ij}^{(k)}/b_{ij}^{(k)})$ is a σ_y-remainder with ϕ_{α_i,β_i} being the map for (α_i, β_i)-shift reduction. By letting $\tilde{a}_{ij}^{(k)} = \phi_{\alpha_i,\beta_i}(a_{ij}^{(k)}) \in \mathbb{F}[y, z]$, one sees from definition that $\deg_y(\tilde{a}_{ij}^{(k)}) < \deg_y(\phi_{\alpha_i,\beta_i}(b_{ij}^{(k)})) = \deg_y(b_{ij}^{(k)})$ and $a_{ij}^{(k)} = \phi_{\alpha_i,\beta_i}^{-1}(\tilde{a}_{ij}^{(k)}) = \tilde{a}_{ij}^{(k)}(\beta_i^{-1} y, \beta_i z + \alpha_i y)$. It follows that every $a_{ij}^{(k)}$ can be viewed as a polynomial in $Z_i = \alpha_i y + \beta_i z$ with coefficients all having degrees in y less than $\deg_y(b_{ij}^{(k)})$. In this case, rather than naively considering the coefficients with respect to y and z, we instead force all the coefficients with respect to y and Z_i of the numerator of $\sum_{k=0}^{\ell} u_k \cdot (a_{ij}^{(k)}/b_{ij}^{(k)})$ to zero. This way ensures that the resulting linear system over $\mathbb{F}$ typically has smaller size than the naive one.

6. Implementation and timings

We have implemented our new algorithm **ReductionCT** in the computer algebra system Maple 2018. Our implementation includes the two enhancements to step 4 discussed in the previous subsection. In order to get an idea about the efficiency of our algorithm, we applied our implementation to certain examples and tabulated their runtime in this section. All timings were measured in seconds on a Linux computer with 128 GB RAM and fifteen 1.2 GHz Dual core processors. The computations for the experiments did not use any parallelism.

We considered trivariate rational functions of the form

$$f(x, y, z) = \frac{a(x, y, z)}{d_1(x, y, z) \cdot d_2(x, y, z)}, \tag{6.1}$$

Table 1
Timings for four variants of the algorithm **ReductionCT**.

(m, n, ξ, ζ)	RCT_1	RCT_2	RCTLM_1	RCTLM_2	order
(1, 1, 1, 1)	0.196	0.098	0.220	0.110	1
(1, 1, 1, 5)	7.319	0.123	9.483	0.123	1
(1, 1, 1, 9)	105.548	0.121	104.514	0.125	1
(1, 1, 1, 13)	2586.295	0.136	3078.043	0.126	1
(1, 1, 1, 3)	0.574	0.097	0.712	0.104	1
(1, 2, 1, 3)	17.812	0.256	17.299	0.263	1
(1, 3, 1, 3)	266.206	1.999	220.209	1.997	1
(1, 4, 1, 3)	2838.827	37.358	3039.199	30.547	1
(1, 5, 1, 3)	19403.916	1074.295	18309.000	1119.393	1
(2, 3, 1, 3)	31678.706	2.540	15825.876	2.224	3
(3, 3, 1, 3)	44243.254	5.378	16869.097	4.295	3
(3, 2, 1, 3)	710.810	0.492	670.501	0.487	3
(3, 2, 2, 3)	1314.809	0.701	941.009	0.756	6
(3, 2, 4, 3)	1558.440	1.525	1121.624	1.550	12
(3, 2, 8, 3)	1878.424	4.215	986.017	4.245	24
(3, 2, 16, 3)	2800.050	21.136	1317.603	38.504	48

where

- $a \in \mathbb{Z}[x, y, z]$ of total degree $m \geq 0$ and max-norm $||a||_\infty \leq 5$, in other words, the maximal absolute value of the coefficients of a with respect to x, y, z are no more than 5;
- $d_i = p_i \cdot \sigma_x^\xi(p_i)$ with $p_1 = P_1(\xi y - \zeta x, \xi z + \zeta x)$ and $p_2 = P_2(\zeta x + \xi y + 2\xi z)$ for two nonzero integers ξ, ζ and two integer polynomials $P_1(y, z) \in \mathbb{Z}[y, z]$, $P_2(z) \in \mathbb{Z}[z]$, both of which have total degree $n > 0$ and max-norm no more than 5.

For a selection of random rational functions of this type for different choices of (m, n, ξ, ζ), Table 1 collects the timings of four variants of the algorithm **ReductionCT** from Section 5. For the column RCT_1, we computed both the telescoper and the certificate, and for the column RCT_2 only the telescoper is computed. The difference between these two variants mainly lies in the time used to bring the certificate to a common denominator. When it is acceptable to keep the certificate as an unnormalized linear combination of rational functions, the timings are virtually the same as for RCT_2. For columns RCTLM_1 and RCTLM_2, we perform the same functionality as RCT_1 and RCT_2 but using the LCLM version of the algorithm **ReductionCT**. Note that the computation of the least common left multiples therein was accomplished by the built-in Maple command OreTools[LCM]['left']. We remark that the performance of the LCLM version of the algorithm **ReductionCT** deteriorates for larger examples, especially when there are many shift equivalence classes in the denominator of the input rational function or the order of a minimal telescoper is relatively high.

We have also compared our procedures with the two Mathematica packages: HolonomicFunctions by Koutschan [39] and MultiSum[1] by Wegschaider (substantially improved by Riese) [47,42]. The HolonomicFunctions, to our best knowledge, is the most comprehensive implementation in terms of creative telescoping for holonomic functions (cf. [37, §2.2]) in more than two variables. There are two commands available in the package for our purpose. One is called **CreativeTelescoping**, which implements Chyzak's algorithm [28] for single sums and can be applied iteratively to compute telescopers for trivariate rational functions. The other is called **FindCreativeTelescoping**, which is based on Koutschan's heuristic approach [38] and constructs the telescoper directly by guessing the denominators of the certificate, as well as their numerator degrees, and solving a linear system. The MultiSum extends the multivariate version of "Sister Celine's technique" developed by Wilf and Zeilberger [48]. The available command in the package is called **FindRecurrence**, which finds a telescoper and a corresponding certificate for a given summand only if the structure set, which is usually not known in advance, is chosen in a clever way. The idea employed in the package is to use random parameter substitutions to quickly rule out useless structure sets, which however requires a priori bounds for the shifts involved (see [42] for further details). We remark that[2] it would be interesting to see in the future if our fully automatic method could provide these extra bounds also automatically, and then the combination of the two methods might yield even a new fully automatic (and efficient) method.

Experiments suggest a better performance of our algorithm. For example, for the rational function

$$f = \frac{4x+2}{(45x+5y+10z+47)(45x+5y+10z+2)(63x-5y+2z+58)(63x-5y+2z-5)}$$

which was constructed using (6.1) with parameter $(m, n, \xi, \zeta) = (1, 1, 1, 9)$, our algorithm found a minimal telescoper for f along with its corresponding certificate in about 3 minutes; while the command **FindRecurrence**, along with a priori bounds $1, 9, 9$ for the shifts in x, y, z, respectively, accomplished the same job using about 7 minutes, the command **CreativeTelescoping** took about 4 hours, and the command **FindCreativeTelescoping** did not finish in reasonable time, which happens because the guessed denominators are wrong/insufficient, and therefore the command finds nothing and runs forever. The same phenomenon was observed for larger examples.

7. Conclusion and future work

In this paper, we presented a new creative telescoping algorithm for the class of trivariate rational functions. The procedure is based on a bivariate extension of Abramov's reduction method initiated in [1]. Our algorithm finds a minimal telescoper for a given

[1] We thank the anonymous referee for bringing this package to our attention.
[2] We thank the anonymous referee for pointing this out.

trivariate rational function without also needing to compute an associated certificate. A Maple implementation indicates the efficiency of our algorithm. As a next step, we are going to investigate the theoretical complexity of our algorithm to see if it matches with the practical performance, something briefly alluded to in the introduction.

We are interested in the more general and important problem of computing hypergeometric multiple summations or proving identities which involve such summations. A function $f(x, y_1, \ldots, y_n)$ is called a multivariate *hypergeometric term* if the quotients

$$\frac{f(x+1, y_1, \ldots, y_n)}{f(x, y_1, \ldots, y_n)}, \frac{f(x, y_1+1, \ldots, y_n)}{f(x, y_1, \ldots, y_n)}, \ldots, \frac{f(x, y_1, \ldots, y_n+1)}{f(x, y_1, \ldots, y_n)}$$

are all rational functions in $x, y_1, \ldots, y_n$. The problem of hypergeometric multiple summations tends to appear more often than the rational case, particularly in combinatorics [11,18], and it is also more challenging.

Since a large percent of hypergeometric terms falls into the class of holonomic functions, the problem of hypergeometric multiple summations can also be considered in a more general framework of multivariate holonomic functions. In this context, several creative telescoping approaches have already been developed in [49,46,29,28,38,13]. The algorithms in the first three papers are based on elimination and suffer from the disadvantage of inefficiency in practice. The algorithm in [28], also known as Chyzak's algorithm, deals with single sums (and single integrals) and can only be used to solve multiple ones in an iterative manner. A fast but heuristic approach was given in [38] in order to eliminate the bottleneck in Chyzak's algorithm of solving a coupled first-order system. This approach generalizes to multiple sums (and multiple integrals). We refer to [37] for a detailed and excellent exposition of these approaches. The work in [13] describes even a general framework that unities the difference ring and the holonomic approach. We remark that all these approaches find the telescoper and the certificate simultaneously, with the exception of Takayama's algorithm in [46] where natural boundaries have to be assured a priori. Note also that holonomicity is a sufficient but not necessary condition for the applicability of creative telescoping applied to hypergeometric terms (cf. [5,23]).

Restricted to the hypergeometric setting, partial solutions for the problem of multiple summations were proposed in [27] and [18]. In the former paper, the authors presented a heuristic method to find telescopers for trivariate hypergeometric terms, through which they also managed to prove certain famous hypergeometric double summation identities. In the latter paper, the authors mainly focused on a subclass of hypergeometric summations – multiple binomial sums. They first showed that the generating function of a given multiple binomial sum is always the diagonal of a rational function and vice versa. They then constructed a differential equation for the diagonal by a reduction-based telescoping approach. Finally the differential equation is translated back into a recurrence relation satisfied by the given binomial sum. In the future, we hope to explore this topic further and aim at developing a complete reduction-based telescoping algorithm for hypergeometric terms in three or more variables.

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