Assignment 3 Solutions

Task 1(a)

(a) $f = [1\ 2\ 3\ 4]$; $\boxed{N=4}$

DFT: $F_k = \frac{1}{N}\sum_{n=0}^{N-1} f_n \overline{\omega}^{nk}$ where $\overline{\omega} = e^{-i\frac{2\pi}{4}}$

$= e^{-i\frac{\pi}{2}} = -i$

$F_0 = \frac{1}{4}[f_0 + f_1 + f_2 + f_3] = \frac{1}{4}[10] = \boxed{2.5}$

$F_1 = \frac{1}{4}[f_0\overline{\omega}^0 + f_1\overline{\omega}^1 + f_2\overline{\omega}^2 + f_3\overline{\omega}^3]$

$= \frac{1}{4}[1 + 2(-i) + 3(-i)^2 + 4(-i)^3]$

$= \frac{1}{4}[1 - 2i - 3 + 4i] = \frac{1}{4}[-2 + 2i]$

$= \boxed{-0.5 + 0.5i}$

$F_2 = \frac{1}{4}[f_0(\overline{\omega}^2)^0 + f_1(\overline{\omega}^2)^1 + f_2(\overline{\omega}^2)^2 + f_3(\overline{\omega}^2)^3]$

$= \frac{1}{4}[1 + 2(-1)^1 + 3(-1)^2 + 4(-1)^3]$

$= \frac{1}{4}[1 - 2 + 3 - 4] = \frac{1}{4}[-2] = \boxed{-0.5}$

$F_3 = \frac{1}{4}[f_0(\overline{\omega}^3)^0 + f_1(\overline{\omega}^3)^1 + f_2(\overline{\omega}^3)^2 + f_3(\overline{\omega}^3)^3]$

$= \frac{1}{4}[1 + 2(i) + 3(i)^2 + 4(i)^3]$

$= \frac{1}{4}[1 + 2i - 3 - 4i] = \frac{1}{4}[-2 - 2i] = \boxed{-0.5-0.5i}$

Therefore: $\mathbf{F} = [F_0\ F_1\ F_2\ F_3]$

$= [2.5\quad -0.5+0.5i\quad -0.5\quad -0.5-0.5i]$

Task 1 (b)

$\{f_n\}_{n=0}^{N-1}$ are real; $N = 2^m$; $f_{N+n} = f_n$ (i.e. periodic data)

<u>To prove:</u> If $f_n = f_{N-n}$ then the Fourier coefficients $\{F_k\}$ are all real.

<u>Proof:</u>

$$F_k = \frac{1}{N} \sum_{n=0}^{N-1} f_n \overline{W}^{nk} \qquad \text{where } W = e^{i\frac{2\pi}{N}}$$

$$= \frac{1}{N} \left[\sum_{n=0}^{\frac{N}{2}-1} f_n \overline{W}^{nk} + \sum_{n=\frac{N}{2}}^{N-1} f_n \overline{W}^{nk} \right]$$

$$= \frac{1}{N} \left[\sum_{n=0}^{\frac{N}{2}-1} f_n \overline{W}^{nk} + \sum_{n=1}^{\frac{N}{2}} f_{N-n} \overline{W}^{(N-n)k} \right]$$

by change of index of summation $n \to N-n$.

Note that $\overline{W}^N = 1$ (so $\overline{W}^{Nk} = 1$) so:

$$F_k = \frac{1}{N} \left[f_0 + \sum_{n=1}^{\frac{N}{2}-1} f_n \overline{W}^{nk} + \sum_{n=1}^{\frac{N}{2}-1} f_{N-n} \overline{W}^{-nk} + f_{\frac{N}{2}} \overline{W}^{-\frac{N}{2}k} \right].$$

Now $\overline{W}^{-\frac{N}{2}} = \overline{W}^{\frac{N}{2}} = e^{i\pi}$ (or $e^{-i\pi}$) $= -1$, so:

$$F_k = \frac{1}{N} \left[f_0 + \sum_{n=1}^{\frac{N}{2}-1} f_n \left(\overline{W}^{nk} + \overline{W}^{-nk} \right) + f_{\frac{N}{2}} (-1)^k \right]$$

because $f_{N-n} = f_n$.

But each term here is <u>real</u> because ($\overline{W}^{nk}$ + its conjugate) is real.

Task 1(c)

To prove: $\displaystyle\sum_{k=0}^{N-1} F_k \bar{F}_k = \frac{1}{N}\sum_{n=0}^{N-1} f_n \bar{f}_n$.

Proof:

$$\sum_{k=0}^{N-1} F_k \bar{F}_k = \sum_{k=0}^{N-1}\left(\frac{1}{N}\sum_{n=0}^{N-1} f_n \bar{W}^{nk}\right)\left(\frac{1}{N}\sum_{m=0}^{N-1} \bar{f}_m W^{mk}\right)$$

(**Note:** Use different indices of summation, n and m, otherwise you will have confusion!)

$$= \frac{1}{N^2}\sum_{k=0}^{N-1}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1} f_n \bar{f}_m W^{-nk} W^{mk} .$$

Move the summation indexed by k to the inside:

$$= \frac{1}{N^2}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1} f_n \bar{f}_m \sum_{k=0}^{N-1} W^{(m-n)k} ,$$

But $\displaystyle\sum_{k=0}^{N-1}\left(W^{m-n}\right)^k = \begin{cases} N, & \text{if } m=n \\ 0, & \text{otherwise} \end{cases}$ so:

$$\sum_{k=0}^{N-1} F_k \bar{F}_k = \frac{1}{N^2}\sum_{n=0}^{N-1} f_n \bar{f}_n N$$

because only the term $m=n$ is nonzero in the summation over m.

i.e. $\displaystyle\sum_{k=0}^{N-1} F_k \bar{F}_k = \frac{1}{N}\sum_{n=0}^{N-1} f_n \bar{f}_n$.