A2, Question 2

We are to write a Matlab function

$$[a, b, c] = MySpline(x, y)$$

where

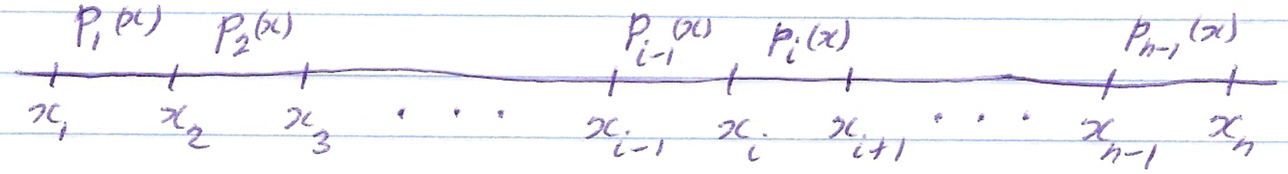
x, y - input vectors of (x_i, y_i) data $(1 \le i \le n)$

a, b, c - output vectors of the coefficients of the natural cubic spline which interpolates (x, y), using the <u>special</u> <u>representation</u>:

$$p_i(x) = a_{i-1} \frac{(x_{i+1} - x)^3}{6 h_i} + a_i \frac{(x - x_i)^3}{6 h_i} + b_i (x_{i+1} - x) + c_i (x - x_i),$$

$$\left[\text{where } h_i = x_{i+1} - x_i\right], \qquad \text{for } i = 1, \ldots, n-1.$$

Schematically,



In order to solve for the coefficients $\{a_i\}_{i=0}^{n-1}$, $\{b_i\}_{i=1}^{n-1}$ and $\{c_i\}_{i=1}^{n-1}$, apply the cubic spline conditions to $p_i(x)$, $1 \le i \le n-1$.

We will need to use the ~~the~~ derivative formulas:

$$p_i'(x) = -3 a_{i-1} \frac{(x_{i+1} - x)^2}{6 h_i} + 3 a_i \frac{(x - x_i)^2}{6 h_i} - b_i + c_i$$

$$p_i''(x) = a_{i-1} \frac{(x_{i+1} - x)}{h_i} + a_i \frac{(x - x_i)}{h_i}$$

Interpolation conditions:

① $p_i(x_i) = y_i \implies \frac{1}{6} a_{i-1} h_i^2 + b_i h_i = y_i$, for $i = 1, \ldots, n-1$

② $p_i(x_{i+1}) = y_{i+1} \implies \frac{1}{6} a_i h_i^2 + c_i h_i = y_{i+1}$, for $i = 1, \ldots, n-1$

Note that if we know a_i's then we can solve equation ① for b_i and equation ② for c_i:

$$\boxed{b_i = \frac{y_i}{h_i} - \frac{1}{6} a_{i-1} h_i}; \qquad \boxed{c_i = \frac{y_{i+1}}{h_i} - \frac{1}{6} a_i h_i};$$

for $i = 1, \ldots, n-1$.

1st derivative conditions

$p_i'(x_{i+1}) = p_{i+1}'(x_{i+1})$, for $i = 1, \ldots, n-2$

$$\implies 3 a_i \frac{h_i^2}{6 h_i} - b_i + c_i = -3 a_i \frac{h_{i+1}^2}{6 h_{i+1}} - b_{i+1} + c_{i+1}$$

$$\implies \frac{1}{2} a_i h_i - b_i + c_i = -\frac{1}{2} a_i h_{i+1} - b_{i+1} + c_{i+1}$$

Plugging in the above formulas for b_i and c_i in terms of a_i's we get, after rearranging terms to put terms involving a_i's on the left hand side:

$$\frac{1}{6} h_i a_{i-1} + \frac{1}{3}(h_i + h_{i+1}) a_i + \frac{1}{6} h_{i+1} a_{i+1} = r_i, \quad i = 1, \ldots, n-2$$

where the right hand side values are

$$r_i = \frac{(y_{i+2} - y_{i+1})}{h_{i+1}} - \frac{(y_{i+1} - y_i)}{h_i}.$$

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Hint: for Question 3 (continued)

We now have a linear system of $n-2$ equations in the n unknowns $a_i, i = 0, 1, ..., n-1$. We must add two boundary conditions before we can solve the system for a unique solution.

Note that the linear system is *tridiagonal*, and we can maintain this property if we add the two "arbitrary" additional conditions in the following form. Let the *first* equation be

$$t_0 a_0 + t_1 a_1 = r_0$$

and let the *last* equation be

$$t_2 a_{n-2} + t_3 a_{n-1} = r_{n-1}$$

where $t_0, t_1, t_2, t_3, r_0, r_{n-1}$ are constants to be chosen, depending on the desired boundary conditions.

We can now express the $n \times n$ tridiagonal linear system in the form

$$Ta = r$$

where a is the vector of coefficients $a_i, i = 0, 1, ..., n-1$ to be solved for, r is the vector of right-hand-side quantities, and T is a tridiagonal matrix.

Now for Question 3, we are asked to compute the coefficients for the *natural cubic spline* so you must choose the "arbitrary" parameters in the first and last equations so as to satisfy the so-called *natural* boundary conditions.

Second derivative conditions: Finally, you should verify that the continuity conditions for the second derivative are satisified.