Inversion of Toeplitz Matrices With Only Two Standard Equations

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ABSTRACT

It is shown that the invertibility of a Toeplitz matrix can be determined through the solvability of two standard equations. The inverse matrix is represented by two of its columns (which are the solutions of the two standard equations) and the entries of the original Toeplitz matrix.

1. INTRODUCTION

Let A be an n-by-n Toeplitz matrix:

$$A = \begin{bmatrix} a_0 & a_{-1} & a_{-2} & \cdots & a_{-(n-1)} \\ a_1 & a_0 & a_{-1} & \cdots & a_{-(n-2)} \\ a_2 & a_1 & a_0 & \cdots & a_{-(n-3)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_0 \end{bmatrix},$$

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where $a_{-(n-1)}, \ldots, a_{n-1}$ are complex numbers. We use the shorthand $A = (a_{p-q})_{p,q=1}^{n}$ for a Toeplitz matrix.

The results of Gohberg and Semencul [3] show that the inverse of a regular Toeplitz matrix can be sometimes represented via its first and last columns. Their theorem is as follows:

THEOREM 1.1 (Gohberg and Semencul). *Let $A = (a_{p-q})_{p,q=1}^{n}$ be a Toeplitz matrix. If each of the systems of equations*

$$\sum_{q=1}^{n} a_{p-q} x_q = \delta_{p,1} \qquad (p = 1, 2, \ldots, n),$$

$$\sum_{q=1}^{n} a_{p-q} y_q = \delta_{p,n} \qquad (p = 1, 2, \ldots, n)$$

is solvable and $x_1 \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{x_1}\left(\begin{bmatrix} x_1 & 0 & \cdots & 0 \\ x_2 & x_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ x_n & x_{n-1} & \cdots & x_1 \end{bmatrix}\begin{bmatrix} y_n & y_{n-1} & \cdots & y_1 \\ 0 & y_n & \cdots & y_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & y_n \end{bmatrix}\right.$$

$$\left. - \begin{bmatrix} 0 & \cdots & 0 & 0 \\ y_1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ y_{n-1} & \cdots & y_1 & 0 \end{bmatrix}\begin{bmatrix} 0 & x_n & \cdots & x_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_n \\ 0 & 0 & \cdots & 0 \end{bmatrix}\right).$$

Another case in which two columns of the inverse of a regular Toeplitz matrix are sometimes sufficient to represent the entire inverse matrix appears in [2].

THEOREM 1.2 (Gohberg and Krupnik). *Let $A = (a_{p-q})_{p,q=1}^{n}$ be a Toeplitz matrix. If each of the systems of equations*

$$\sum_{q=1}^{n} a_{p-q} x_q = \delta_{p,1} \qquad (p = 1, 2, \ldots, n),$$

$$\sum_{q=1}^{n} a_{p-q} z_q = \delta_{p,2} \qquad (p = 1, 2, \ldots, n)$$

is solvable and $x_n \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{x_n}\left(\begin{bmatrix} z_1 & 0 & \cdots & 0 \\ z_2 & z_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ z_n & z_{n-1} & \cdots & z_1 \end{bmatrix}\begin{bmatrix} 0 & x_n & \cdots & x_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_n \\ 0 & 0 & \cdots & 0 \end{bmatrix}\right.$$

$$-\begin{bmatrix} x_1 & 0 & \cdots & 0 \\ x_2 & x_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ x_n & x_{n-1} & \cdots & x_1 \end{bmatrix}\begin{bmatrix} 0 & z_n & \cdots & z_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & z_n \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$\left.+\begin{bmatrix} x_1x_n & x_1x_{n-1} & \cdots & x_1x_1 \\ x_2x_n & x_2x_{n-1} & \cdots & x_2x_1 \\ \vdots & \vdots & \ddots & \vdots \\ x_nx_n & x_nx_{n-1} & \cdots & x_nx_1 \end{bmatrix}\right). \tag{1.1}$$

However, this is not always the case. It is shown in [2] that the inverse of the Toeplitz matrix

$$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

cannot be determined by any pair of its columns.

Ben-Artzi and Shalom have shown in [1] that three columns of the inverse of a regular Toeplitz matrix are always enough to reconstruct it.

THEOREM 1.3 (Ben-Artzi and Shalom). *Let $A = (a_{p-q})_{p,q=1}^n$ be a Toeplitz matrix. If each of the systems of equations*

$$\sum_{q=1}^{n} a_{p-q}x_q = \delta_{p,1} \qquad (p = 1,2,\dots,n),$$

$$\sum_{q=1}^{n} q_{p-q}y_q = \delta_{p,n+1-l} \;(p = 1,2,\dots,n),$$

$$\sum_{q=1}^{n} a_{p-q}z_q = \delta_{p,n+2-l} \;(p = 1,2,\dots,n),$$

is solvable for any integer number l $(1 \leqslant l \leqslant n)$ for which $x_l \neq 0$, then A is invertible with inverse given by

$$\frac{1}{x_l}\left(\begin{bmatrix} x_1 & 0 & \cdots & 0 \\ x_2 & x_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ x_n & x_{n-1} & \cdots & x_1 \end{bmatrix}\begin{bmatrix} y_n & y_{n-1} - z_n & \cdots & y_1 - z_2 \\ 0 & y_n & \cdots & y_2 - z_3 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & y_n \end{bmatrix}\right.$$
$$\left.+\begin{bmatrix} z_1 & \cdots & 0 & 0 \\ z_2 - y_1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ z_n - y_{n-1} & \cdots & z_2 - y_1 & z_1 \end{bmatrix}\begin{bmatrix} 0 & x_n & \cdots & x_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_n \\ 0 & 0 & \cdots & 0 \end{bmatrix}\right). \tag{1.2}$$

The results of [8] show that the inverse of a Toeplitz matrix A can always be determined by only two systems of equations having A as coefficient matrix. Subsequent results of [4], [7], and [6] show that one of the two equations can be chosen to be of the form

$$AX = E, \tag{1.3}$$

where E is a column vector in the standard basis in $\mathbb{C}^n$. In all four cases, however, the second equation has a right-hand side that depends on the entries of the matrix A.

The main result of this paper shows that it is enough to solve only two systems of linear equations of the form (1.3) where E is a column vector in the standard basis in $\mathbb{C}^n$ in order to determine the invertibility of A. In this case, we obtain a formula for A^{-1} in terms of two of its columns and the entries of A.

THEOREM 1.4. *The Toeplitz matrix $A = (a_{p-q})_{p,q=1}^n$ is invertible if and only if the following conditions hold:*

(a) *There exists a solution for the following system of equations:*

$$\sum_{q=1}^{n} a_{p-q} x_q = \delta_{p,1} \qquad (p = 1, 2, \dots, n).$$

(b) *Let l be such that $x_l \neq 0$ and $x_q = 0$ for all $q > l$. Then there exists a solution for the following system of equations:*

$$\sum_{q=1}^{n} a_{p-q} z_q = \delta_{p,n+2-l} \qquad (p = 1, 2, \dots, n).$$

In case $l = n$ then A^{-1} is given by (1.1), *while in case $l < n$ then A^{-1} is given by* (1.2) *for*

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \left(S - \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \begin{bmatrix} a_{-1} & a_{-2} & \cdots & a_{-(n-1)} & 0 \end{bmatrix} \right)^{n-l} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}. \quad (1.4)$$

Here S denotes the n-by-n lower shift matrix

$$S = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & & \ddots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

In the next section we prove Theorem 1.4 and obtain some corollaries. In Section 3 we state the analogous results for Hankel matrices, and in Section 4 we consider the issue of minimality. Namely, when can the solution of a unique system of linear equations $AX = E$ determine the invertibility of the Toeplitz (or Hankel) matrix A?

2. PROOF OF THE MAIN RESULT

First let us present some useful notation. We denote the row with entries $b_1, b_2, \dots, b_s$ either by $\mathrm{row}(b_1, b_2, \dots, b_s)$ or by $\mathrm{row}(b_j)_{j=1}^s$. The column with entries $b_1, b_2, \dots, b_s$ is denoted either by $\mathrm{col}(b_1, b_2, \dots, b_s)$ or by $\mathrm{col}(b_i)_{i=1}^s$. Let $E^{(1)}, E^{(2)}, \dots, E^{(n)}$ and $F^{(1)}, F^{(2)}, \dots, F^{(n)}$ be the n unit columns and unit rows, i.e.,

$$E^{(q)} = \mathrm{col}(\delta_{p,q})_{p=1}^n \qquad (q = 1, 2, \dots, n)$$

and

$$F^{(q)} = \mathrm{row}(\delta_{p,q})_{p=1}^n \qquad (q = 1, 2, \dots, n).$$

We extend the definition in the sense that $E^{(0)}$, for example, is the zero column.

For any matrix A we denote its transposed matrix by A^T.

LEMMA 2.1. *Let* $A = (a_{p-q})_{p,q=1}^{n}$ *be a Toeplitz matrix and* $X = \mathrm{col}(x_p)_{p=1}^{n}$ *be a column vector for which*

$$AX = E^{(1)}. \tag{2.1}$$

Define the column vectors

$$X^{(i)} = (S - XF^{(1)}AS)^{i} X \qquad (i \geqslant 0). \tag{2.2}$$

Note that in particular $X^{(0)} = X$. *Let* l *be such that* $x_l \neq 0$ *and* $x_q = 0$ *for all* $q > l$, *and denote* $X^{(i)} = \mathrm{col}(x_p^{(i)})_{p=1}^{n}$. *Then*

$$AX^{(i)} = E^{(i+1)} \qquad (i = 0, 1, \dots, n - l)$$

and

$$x_p^{(i)} = 0 \qquad (i = 0, 1, \dots, n - l, \quad p > l + i).$$

Proof. We prove the assertions by induction on i. The case of $i = 0$ follows from the assumption on X. Next, assume that the lemma is valid for $i < n - l$, and prove it for $i + 1$. Note that (2.2) implies

$$X^{(i+1)} = (S - XF^{(1)}AS) X^{(i)}, \tag{2.3}$$

and therefore,

$$x_p^{(i+1)} = F^{(p)}X^{(i+1)} = x_{p-1}^{(i)} - x_p F^{(1)}ASX^{(i)}.$$

But if $p > l + i + 1$, then in particular $x_p = 0$, and also $x_{p-1}^{(i)} = 0$ (as $p - 1 > l + i$). Therefore,

$$x_p^{(i+1)} = 0.$$

Moreover, it follows from (2.1) and (2.3) that

$$AX^{(i+1)} = (AS - E^{(1)}F^{(1)}AS) X^{(i)}.$$

But

$$SA - AS = SAE^{(n)}F^{(n)} - E^{(1)}F^{(1)}AS$$

for any n-by-n Toeplitz matrix A, so

$$AX^{(i+1)} = (SA - SAE^{(n)}F^{(n)})X^{(i)}.$$

Note that $F^{(n)}X^{(i)} = x_n^{(i)} = 0$, as we assume that $i < n - l$. Furthermore, $SAX^{(i)} = SE^{(i)} = E^{(i+1)}$. ■

Proof of Theorem 1.4. It is clear that if the Toeplitz matrix A is invertible, then both systems of equations are solvable. Conversely, assume that both conditions are fulfilled. In particular, (2.1) holds for the column vector $X = \operatorname{col}(x_p)_{p=1}^n$. Thus, it follows from Lemma 2.1 that

$$AY = E^{(n+1-l)}$$

for the column vector

$$Y = (S - XF^{(1)}AS)^{n-l}X. \tag{2.4}$$

Furthermore, condition (b) implies the existence of a column vector Z [which is precisely $\operatorname{col}(z_p)_{p=1}^n$] for which

$$AZ = E^{(n+2-l)}.$$

It follows from Theorem 1.3 that the matrix A is indeed invertible. The formula for A^{-1} is a simple consequence of the identity between (2.4) and (1.4). ■

In Theorem 1.4 there are two extreme cases. The first one is when $l = n$. This is precisely the case that appeared in [2] and is stated as Theorem 1.2. The second case is when $l = 1$, which happens when the matrix is an upper triangular Toeplitz matrix. This result can be easily proved independently of our main result. However, we state it for illustrative purposes.

COROLLARY 2.2. *Let $A = (a_{p-q})_{p,q=1}^n$ be a Toeplitz matrix. There exists a complex number x such that*

$$A \cdot [xE^{(1)}] = E^{(1)}$$

if and only if A is an upper triangular matrix (i.e., $a_i = 0$ for $i > 0$) and $a_0 \neq 0$. In this case, $x = 1/a_0$, the matrix A is invertible, and its inverse *matrix A^{-1} is the upper triangular Toeplitz matrix*

$$A^{-1} = \begin{bmatrix} y_n & y_{n-1} & \cdots & y_1 \\ 0 & y_n & \cdots & y_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & y_n \end{bmatrix},$$

in which

$$\mathrm{col}(y_p)_{p=1}^n = \frac{1}{a_0} \begin{bmatrix} -a'_{-1} & -a'_{-2} & \cdots & -a'_{-(n-1)} & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}^{n-1} E^{(1)},$$

where $a'_{-i} = a_{-i}/a_0$ for $i = 1, 2, \dots, n-1$.

Clearly, an analogous result holds for lower triangular Toeplitz matrices as well. In general, for regular Toeplitz matrices the last column of the inverse can replace the first one. Also in this case, we might need an additional column.

THEOREM 2.3. *The Toeplitz matrix $A = (a_{p-q})_{p,q=1}^n$ is invertible if and only if the following conditions hold:*

(a) *There exists a solution for $AX = E^{(n)}$.*

(b) *Denote $X = \mathrm{col}(x_p)_{p=1}^n$, and let m $(1 \leqslant m \leqslant n)$ be such that $x_m \neq 0$ and $x_q = 0$ for all $q < m$. Then there exists a solution for $AZ = E^{(n-m)}$.*

In this case, denote $Z = \mathrm{col}(z_p)_{p=1}^n$; then the inverse is given by

$$\frac{1}{x_m}\left(\begin{bmatrix} x_n & x_{n-1} & \cdots & x_1 \\ 0 & x_n & \cdots & x_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_n \end{bmatrix} \begin{bmatrix} y_1 & \cdots & 0 & 0 \\ y_2 - z_1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ y_n - z_{n-1} & \cdots & y_2 - z_1 & y_1 \end{bmatrix}\right.$$

$$\left. + \begin{bmatrix} z_n & z_{n-1} - y_n & \cdots & z_1 - y_2 \\ 0 & z_n & \cdots & z_2 - y_3 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & z_n \end{bmatrix} \begin{bmatrix} 0 & \cdots & 0 & 0 \\ x_1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ x_{n-1} & \cdots & x_1 & 0 \end{bmatrix}\right),$$

where the column $Y = \operatorname{col}(y_p)_{p=1}^n$ *is defined by*

$$Y = \left(S^T - XF^{(n)}AS^T\right)^{m-1}X.$$

Proof. The proof follows essentially from Theorem 1.4, noting that for any Toeplitz matrix A, its transposed matrix A^T (which is clearly a Toeplitz matrix as well) satisfies

$$A = JA^TJ$$

for the n-by-n matrix

$$J = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & \vdots & \cdot^{\cdot^{\cdot}} & 1 & 0 \\ \vdots & 0 & \cdot^{\cdot^{\cdot}} & 0 & \vdots \\ 0 & 1 & \cdot^{\cdot^{\cdot}} & \vdots & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}. \tag{2.5}$$

Now, $J\operatorname{col}(x_p)_{p=1}^n = \operatorname{col}(x_{n+1-p})_{p=1}^n$, so the equation $AX = E^{(n)}$ is equivalent to $(JAJ)(JX) = JE^{(n)}$ (as J^2 is the identity matrix), or to $A^T\operatorname{col}(x_{n+1-p})_{p=1}^n = E^{(1)}$. Note that x_m is the last nonzero entry of X if and only if x_l is the first nonzero entry of JX for $l = n+1-m$. Apply Theorem 2.3 to the Toeplitz matrix A^T, the column JX and $l = n+1-m$. Let $\operatorname{col}(z_{n+1-p})_{p=1}^n$ be the solution of the second system of equations [part (b) of Theorem 1.4]. Namely, $A^TJZ = E^{(n+2-l)}$ for $Z = \operatorname{col}(z_p)_{p=1}^n$. Equivalently, $AZ = E^{(l-1)}$, or $AZ = E^{(n-m)}$. ■

COROLLARY 2.4. *Let* $A = (a_{p-q})_{p,q=1}^n$ *be a Toeplitz matrix. There exists a complex number* x *such that*

$$A \cdot \left[xE^{(n)}\right] = E^{(n)}$$

if and only if A *is a lower triangular matrix* ($i.e.$, $a_i = 0$ *for* $i < 0$) *and* $a_0 \neq 0$. *In this case,* $x = 1/a_0$, *the matrix* A *is invertible, and its inverse matrix* A^{-1} *is the lower triangular Toeplitz matrix*

$$A^{-1} = \begin{bmatrix} y_1 & 0 & \cdots & 0 \\ y_2 & y_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ y_n & y_{n-1} & \cdots & y_1 \end{bmatrix},$$

in which

$$\operatorname{col}(y_p)_{p=1}^{n} = \frac{1}{a_0}\begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & -a'_{n-1} & \cdots & -a'_2 & -a'_1 \end{bmatrix}^{n-1} E^{(n)},$$

where $a'_i = a_i/a_0$ *for* $i = 1, 2, \ldots, n-1$.

As before, the above result is stated for illustrative purposes only, since the formula is easily derived independently of Theorem 2.3.

3. INVERSE FORMULAE FOR HANKEL MATRICES

Let A be an n-by-n Hankel matrix

$$A = \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ a_2 & a_3 & a_4 & \iddots & a_{n+1} \\ a_3 & \vdots & \iddots & \iddots & a_{n+2} \\ \vdots & a_n & \iddots & \iddots & \vdots \\ a_n & a_{n+1} & a_{n+2} & \cdots & a_{2n-1} \end{bmatrix},$$

with $a_1, \ldots, a_{2n-1}$ complex numbers. We use the shorthand $A = (a_{p+q-1})_{p,q=1}^{n}$ to denote these matrices. Hankel matrices have many features in common with Toeplitz matrices. In this section we state and prove the corresponding versions of Theorem 1.4 and 2.3 along with their corollaries as they apply to Hankel matrices.

THEOREM 3.1. *The Hankel matrix* $A = (a_{p+q-1})_{p,q=1}^{n}$ *is invertible if and only if the following conditions hold*:

(a) *There exists a solution for the following system of equations*:

$$\sum_{q=1}^{n} a_{p+q-1} x_{n-q+1} = \delta_{p,1} \qquad (p = 1, 2, \ldots, n).$$

(b) *Let l be such that $x_l \neq 0$ and $x_q = 0$ for all $q > l$. Then there exists a solution for the following system of equations:*

$$\sum_{q=1}^{n} a_{p+q-1} \cdot z_{n-q+1} = \delta_{p,n+2-l} \qquad (p = 1, 2, \dots, n).$$

In addition the inverse is given by

$$\frac{1}{x_l}\left(\begin{bmatrix} x_n & x_{n-1} & \cdots & x_1 \\ \vdots & \vdots & ⋰ & \vdots \\ x_2 & x_1 & \cdots & 0 \\ x_1 & 0 & \cdots & 0 \end{bmatrix}\begin{bmatrix} y_n & y_{n-1} - z_n & \cdots & y_1 - z_2 \\ 0 & y_n & \cdots & y_2 - z_3 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & y_n \end{bmatrix}\right.$$

$$\left. + \begin{bmatrix} z_n - y_{n-1} & \cdots & z_2 - y_1 & z_1 \\ \vdots & ⋰ & \vdots & \vdots \\ z_2 - y_1 & \cdots & 0 & 0 \\ z_1 & \cdots & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & x_n & \cdots & x_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_n \\ 0 & 0 & \cdots & 0 \end{bmatrix}\right)$$

with

$$\operatorname{col}(y_{n-p+1})_{p=1}^{n} = \Big\{S^T - \operatorname{col}(x_{n-p+1})_{p=1}^{n} \cdot [0, a_1, \dots, a_{n-1}]\Big\}^{n-l} \operatorname{col}(x_{n-p+1})_{p=1}^{n}.$$

When $l = n$ the resulting inverse formula is easily transformed into the inversion formula for Hankel matrices given by Gohberg and Krupnik [2]. For the other extreme, namely when $l = 1$, we have:

COROLLARY 3.2. *Let $A = (a_{p+q-1})_{p,q=1}^{n}$ be a Hankel matrix. There exists a complex number x such that*

$$A \cdot [xE^{(n)}] = E^{(1)}$$

if and only if A is an upper quasitriangular matrix (i.e., $a_i = 0$ for $i > n$) and $a_n \neq 0$. In this case, $x = 1/a_n$, the matrix A is invertible, and its inverse matrix A^{-1} is the upper quasitriangular Hankel matrix

$$A^{-1} = \begin{bmatrix} 0 & 0 & \cdots & y_n \\ \vdots & \vdots & ⋰ & \vdots \\ 0 & y_n & \cdots & y_2 \\ y_n & y_{n-1} & \cdots & y_1 \end{bmatrix},$$

in which

$$\begin{bmatrix} y_n \\ \vdots \\ y_2 \\ y_1 \end{bmatrix} = \frac{1}{a_n} \begin{bmatrix} 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \cdot^{\cdot^{\cdot}} & 0 & 0 \\ 0 & 1 & \cdot^{\cdot^{\cdot}} & \vdots & \vdots \\ 1 & 0 & \cdots & 0 & 0 \\ -a'_{n-1} & -a'_{n-2} & \cdots & -a'_1 & 0 \end{bmatrix}^{n-1} E^{(n)},$$

where $a'_i = a_i/a_n$ *for* $i = 1, 2, \ldots, n-1$.

For the corresponding results where the last, rather than the first, column of the inverse is known we have:

THEOREM 3.3. *The Hankel matrix* $A = (a_{p+q-1})_{p,q=1}^n$ *is invertible if and only if the following conditions hold*:

(a) *There exists a solution for the following system of equations*:

$$\sum_{q=1}^{n} a_{p+q-1} x_{n-q+1} = \delta_{p,n} \qquad (p = 1, 2, \ldots, n).$$

(b) *Let* m *be such that* $x_m \neq 0$ *and* $x_q = 0$ *for all* $q < m$. *Then there exists a solution for the following system of equations*:

$$\sum_{q=1}^{n} a_{p+q-1} z_{n-q+1} = \delta_{p,n-m} \qquad (p = 1, 2, \ldots, n).$$

In this case the inverse is given by

$$\frac{1}{x_m}\left(\begin{bmatrix} 0 & 0 & \cdots & x_n \\ \vdots & \vdots & \cdot^{\cdot^{\cdot}} & \vdots \\ 0 & x_n & \cdots & x_2 \\ x_n & x_{n-1} & \cdots & x_1 \end{bmatrix} \begin{bmatrix} y_1 & \cdots & 0 & 0 \\ y_2 - z_1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ y_n - z_{n-1} & \cdots & y_2 - z_1 & y_1 \end{bmatrix} \right.$$

$$\left. + \begin{bmatrix} 0 & 0 & \cdots & z_n \\ \vdots & \vdots & \cdot^{\cdot^{\cdot}} & \vdots \\ 0 & z_n & \cdots & z_2 - y_3 \\ z_n & z_{n-1} - y_n & \cdots & z_1 - y_2 \end{bmatrix} \begin{bmatrix} 0 & \cdots & 0 & 0 \\ x_1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ x_{n-1} & \cdots & x_1 & 0 \end{bmatrix} \right),$$

where

$$\operatorname{col}(y_{n-p+1})_{p=1}^{n}$$

$$= \left\{S - \operatorname{col}(x_{n-p+1})_{p=1}^{n} \cdot [a_{n+1}, \dots, a_{2n-1}, 0]\right\}^{m-1} \operatorname{col}(x_{n-p+1})_{p=1}^{n}.$$

COROLLARY 3.4. *Let $A = (a_{p+q-1})_{p,q=1}^{n}$ be a Hankel matrix. There exists a complex number x such that*

$$A \cdot [xE^{(1)}] = E^{(n)}$$

if and only if A is a lower quasitriangular matrix (i.e., $a_i = 0$ for $i < n$) and $a_n \neq 0$. In this case, $x = 1/a_n$, the matrix A is invertible and its inverse matrix A^{-1} is the lower quasitriangular Hankel matrix

$$A^{-1} = \begin{bmatrix} y_n & y_{n-1} & \cdots & y_1 \\ \vdots & \vdots & .\cdot^{\cdot} & \vdots \\ y_2 & y_1 & \cdots & 0 \\ y_1 & 0 & \cdots & 0 \end{bmatrix},$$

in which

$$\begin{bmatrix} y_n \\ \vdots \\ y_2 \\ y_1 \end{bmatrix} = \frac{1}{a_n} \begin{bmatrix} 0 & -a'_{2n-1} & \cdots & -a'_{n+2} & -a'_{n+1} \\ 0 & 0 & \cdots & 0 & 1 \\ \vdots & \vdots & .\cdot^{\cdot} & 1 & 0 \\ 0 & 0 & .\cdot^{\cdot} & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \end{bmatrix}^{n-1} E^{(1)},$$

where $a'_i = a_i/a_n$ for $i = n+1, n+2, \dots, 2n-1$.

Proof. Theorems 3.1 and 3.3 along with their corollaries are all proved in a similar fashion.

For each $i, j = 1, \dots, n$, define $\hat{a}_i = a_{i-n}$. if J is the n-by-n matrix defined by (2.5), then

$$AJ = (\hat{a}_{p-q})_{p,q=1}^{n} = \hat{A}$$

a Toeplitz matrix. Parts (a) and (b) of Theorem 3.1 then correspond to parts (a) and (b) of Theorem 1.4 applied to the matrix $\hat{A}$. Since J is its own inverse, Theorem 3.1 follows directly from Theorem 1.4 along the observation that

$$A^{-1} = (\hat{A}J)^{-1} = J\hat{A}^{-1}.$$

Similarly, parts (a) and (b) of Theorem 3.3 correspond to parts (a) and (b) of Theorem 2.3 applied to $\hat{A}$. ∎

4. MINIMAL NUMBER OF STANDARD EQUATIONS THAT DETERMINE INVERTIBILITY

In this section, we show that the results stated above are minimal. Namely, one cannot determine the invertibility of a Toeplitz or a Hankel matrix through the solvability of precisely one system of equations of the form

$$AX = E,$$

where E is a unit column vector, unless A is triangular (Toeplitz) or quasitriangular (Hankel).

It has been shown in [5] that one cannot determine the invertibility of an n-by-n Toeplitz matrix A from the existence of solutions to $AX = U_i$ for any $n-1$ fixed vectors $U_1, U_2, \dots, U_{n-1} \in \mathbb{C}^n$. Here we consider a slightly different problem. We are given two unit columns (e.g. $E^{(1)}$ and $E^{(n+2-l)}$ as in Theorem 1.4), and we have shown that the invertibility of A can be deduced, using the structure of the solution columns (in fact, we merely used the fact that x_l was the last nonzero entry of the column X which solves $AX = E^{(1)}$).

THEOREM 4.1. *Let $X = \operatorname{col}(x_p)_{p=1}^n$ be a nonzero column. Let l and m be two integers ($1 \leqslant l, m \leqslant n$) such that $x_l \neq 0$, $x_m \neq 0$, and $x_q = 0$ whenever $q < l$ or $q > m$. Let $E^{(i)}$ be a unit column ($1 \leqslant i \leqslant n$). Then there exists a singular n-by-n Toeplitz matrix $A = (a_{p-q})_{p,q=1}^n$ such that $AX = E^{(i)}$ unless $i = l = m = 1$ or $i = l = m = n$, which occurs if and only if A is a regular traigular Toeplitz matrix (upper in the first case and lower in the second case).*

Proof. First let us consider the case in which $l < i$. Consider the Toeplitz matrix $A = (a_{p-q})_{p,q=1}^n$ in which $a_j = 0$ whenever $j < i - l$, a_{i-l}

$= 1/x_l$, and all the other entries are defined recursively by

$$a_j = \frac{-\sum_{k=1}^{m-l} a_{j-k} x_{l+k}}{x_l} \qquad (j > i - l).$$

This Toeplitz matrix is clearly singular (having zero row as its upper row), and it satisfies $AX = E^{(i)}$.

The second case is of $i < m$. Now we consider the Toeplitz matrix $A = (a_{p-q})_{p,q=1}^n$ in which $a_j = 0$ whenever $j > i - m$, $a_{i-m} = 1/x_m$, and all the other entries are defined recursively by

$$a_j = \frac{-\sum_{k=1}^{m-l} a_{j+k} x_{m-k}}{x_m} \qquad (j < i - m).$$

This Toeplitz matrix is again a singular matrix (having zero row as its lower row) which satisfies $AX = E^{(i)}$.

The last possible case is the one in which $m \leqslant i \leqslant l$. But it is clear that $l \leqslant m$, so in this case $X = xE^{(i)}$ for some nonzero complex number x. If $1 < i < n$, then one can take the Toeplitz matrix $A = (a_{p-q})_{p,q=1}^n$ in which $a_{n-1} = a_0 = a_{-(n-1)} = 1/x$ and $a_j = 0$ whenever $j \neq n-1, 0, -(n-1)$. This singular (the first and last rows are identical) Toeplitz matrix clearly satisfies $A \cdot [xE^{(i)}] = E^{(i)}$.

Note that if $i = 1$ or $i = n$, then the matrix A must be an invertible triangular Toeplitz as discussed in Corollary 2.2 and in Theorem 2.3. ■

For Hankel matrices we have an analogous result, which we state without proof.

THEOREM 4.2. *Let $X = \mathrm{col}(x_{n-p+1})_{p=1}^n$ be a nonzero column. Let l and m be two integers $(1 \leqslant l, m \leqslant n)$ such that $x_l \neq 0$, $x_m \neq 0$, and $x_q = 0$ whenever $q < l$ or $q > m$. Let $E^{(i)}$ be a unit column $(1 \leqslant i \leqslant n)$. Then there exists a singular n-by-n Hankel matrix $A = (a_{p+1-1})_{p,q=1}^n$ such that $AX = E^{(i)}$ unless $i = l = m = 1$ or $i = l = m = n$, which occurs if and only if A is a regular quasitriangular Hankel matrix (upper in the first case and lower in the second case).*

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